On the Palis Conjecture

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Unless stated otherwise, the results are joint with a subset of
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1 Context

We will consider here smooth dynamical systems $f \colon M \to M$ on some smooth compact manifold M of finite dimension.

One of the central aims in dynamical systems is to provide a classification, and perhaps even to show that 'typical systems' don't have pathological behaviour. One obvious classification is the following:

1. f is *hyperbolic* or *Axiom A*.

2. f is not hyperbolic, but f has a *physical measure*. Roughly speaking, this means that for typical starting points x and for continuous observables ϕ, the sequence $\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x))$ converges.

3. other, bad maps.

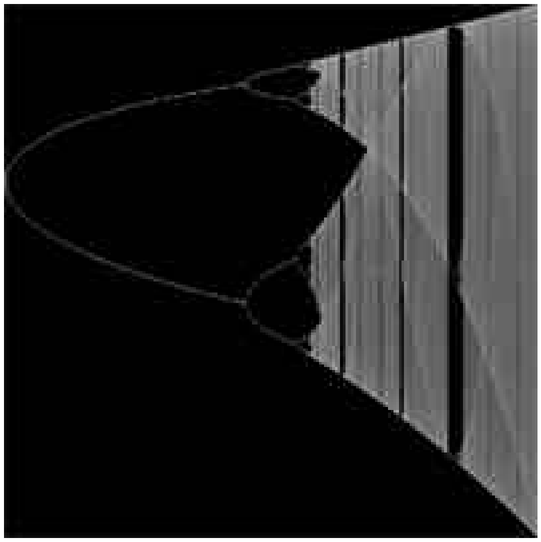
The most ambitious hope would to show that hyperbolic maps are dense. Apparently, up to the late 1960's, Smale believed that hyperbolic systems are dense in all dimensions, but this was shown to be false in the early 1970's for diffeomorphisms on manifolds of dimension ≥ 2 (by Newhouse and others). Whether the space of non-bad maps (in the above sense) is dense is still not known; not even in the C^1 topology.

If $\dim(M) = 1$ much more is known. This is the topic of these talks.

2 Density of Hyperbolicity

The problem of density of hyperbolicity in dimension one goes back in some form to Fatou (in the 1920's). Smale gave this problem 'naively' as a thesis problem in the 1960's (Guckenheimer's thesis). The problem whether hyperbolicity is dense in dimension one was studied by many people, and it was solved in the C^1 topology by Jakobson, a partial solution was given in the C^2 topology by Blokh+Misiurewicz and C^2 density was finally proved by Shen in 2003 or so.

For quadratic maps $f_a = ax(1-x)$, the Fatou conjecture states that the periodic windows are dense in the following diagram (where horizontally a is drawn, and vertically $[0,1]$ and iterates 900-1000 iterates of a typical starting point.



Theorem 1. *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

Here we say that a real one-dimenional map f is *hyperbolic* if each critical point is in the basin of a hyperbolic periodic point. This implies that the real line is the union of a repelling hyperbolic set (a Cantor set of zero Lebesgue measure), the basin of hyperbolic attracting periodic points and the basin of infinity. So for a.e. point x, the iterates $f^n(x)$ tend to a periodic orbit.

In the quadratic case the above theorem says that for a dense set of real parameters a, the non-wandering set of $x \mapsto ax(1-x)$ is hyperbolic. This quadratic case was proved during the mid 90's by Lyubich, Graczyk and Swiatek. But this case is special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.

Note that the above theorem implies that the space of hyperbolic polynomials is an open dense subset in the space of real polynomials of fixed degree. Every hyperbolic map satisfying the mild "no-cycle" condition (critical points are not eventually mapped onto other critical points, are structurally stable).

The above theorem allows us to solve the 2nd part of Smale's eleventh problem for the 21st century.

Theorem 2. *Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \dots, \infty, \omega$.*

Theorem 3. *Any complex polynomial which is not infinitely often renormalizable, can be approximates by a hyperbolic polonomial.*

(If we could prove this without the condition that the map is only finitely renormalizable, the complex Fatou conjecture would follows.)

2.1 Connection with the closing lemma

Theorem 4 (Pugh's Closing Lemma). *Let x be a recurrent point of a smooth diffeomorphism f on a compact manifold. Then there exists a smooth diffeomorphism g which is C^1 close to f for which x is periodic.*

For the last 30 years any attempt to prove the C^k version of this result has been unsuccessful. However, one of the consequences of our result is the one-dimensional version:

Theorem 5 (1-D Closing Lemma). *Assume $k = 1, 2, \ldots, \infty, \omega$ and let x be a recurrent point of a C^k interval map f. Then there exists a smooth map g which is C^k close to f for which x is periodic.*

2.2 Quasi-conformal rigidity

The proof of these result heavily depends on complex analysis. In fact the theorems above can be derived from the following rigidity result.

Theorem 6. *Let f and $\tilde{f}$ be real polynomials of degree n which only have real critical points. If f and $\tilde{f}$ are topologically conjugate (as dynamical systems acting on the real line) and corresponding critical points have the same order, then they are quasiconformally conjugate (on the complex plane).*

(A critical point c is a point so that $f'(c) = 0$. Not all critical points of a real polynomial need to be real.)

If the polynomials are not real, then we need to make an additional assumption:

Theorem 7. *Let f and $\tilde{f}$ be complex polynomials of degree n which are not infinitely renormalizable and only have hyperbolic periodic points. If f and $\tilde{f}$ are topologically conjugate, then they are quasiconformally conjugate.*

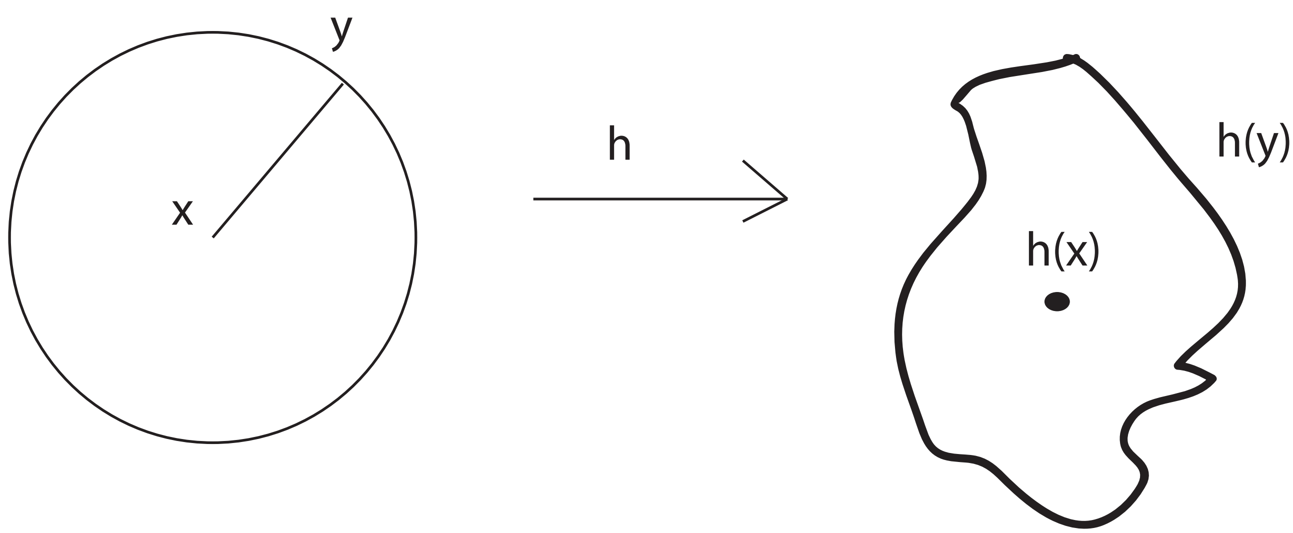
(This generalizes the famous theorem of Yoccoz, proving that the Mandelbrot set associated to the quadratic family $z \mapsto z^2 + c$ is locally connected at non-renormalizable parameters.)

It would be nice to say that the polynomials are C^1 conjugate: but this is obviously wrong: it would imply that eigenvalues of corresponding periodic points are the same. What might be true is the following:

Conjecture: Assume that $f, \tilde{f} \colon \mathbb{R} \to \mathbb{R}$ are C^∞ and conjugate with corresponding critical points of the same (finite) order. If ω is a minimal set for f (i.e., all orbits inside ω are dense in ω), then there exists a C^1 homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ such that $h \circ f = \tilde{f} \circ h$ **restricted to** ω.

The best thing one can hope for is (global) quasiconformally conjugacy: This means that there exist a conjugacy $h \colon \mathbb{C} \to \mathbb{C}$ between $f, \tilde{f} \colon \mathbb{C} \to \mathbb{C}$ and a constant $K < \infty$ such that for Lebesgue almost all $x \in \mathbb{C}$

$$\liminf_{r \to 0} \frac{\sup_{|y-x|=r} |h(y) - h(x)|}{\inf_{|y-x|=r} |h(y) - h(x)|} < K.$$



Such maps are, for example, Hölder and in Lebesgue almost every point they are differentiable (as maps from $\mathbb{C} = \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$).

2.3 How to prove rigidity?

As usual, we shall try to create a Markov map.

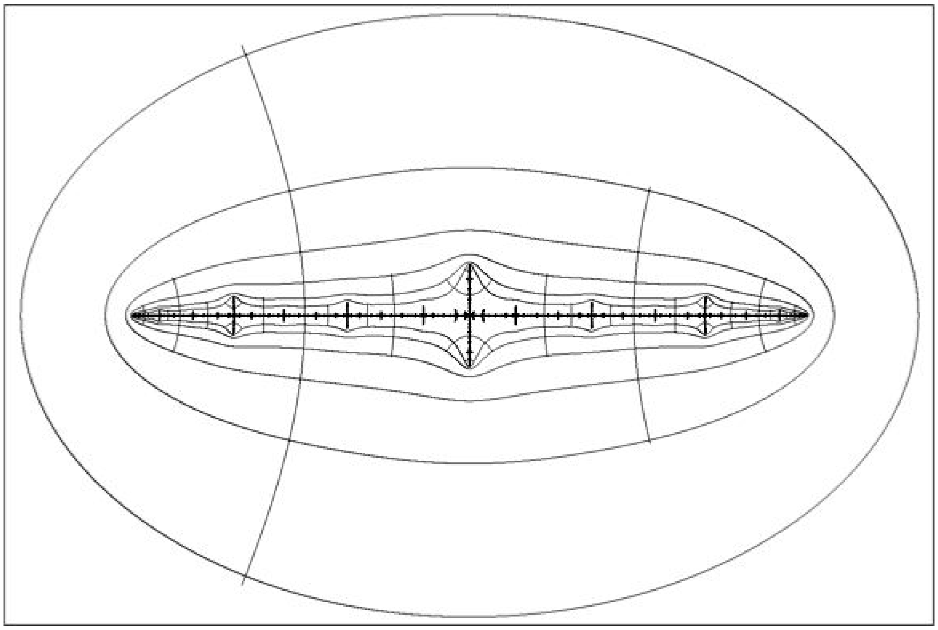
- In the 'weakly hyperbolic' setting one can hope for a countable Bernoulli description: a partition I_i of the relevant dynamical space I

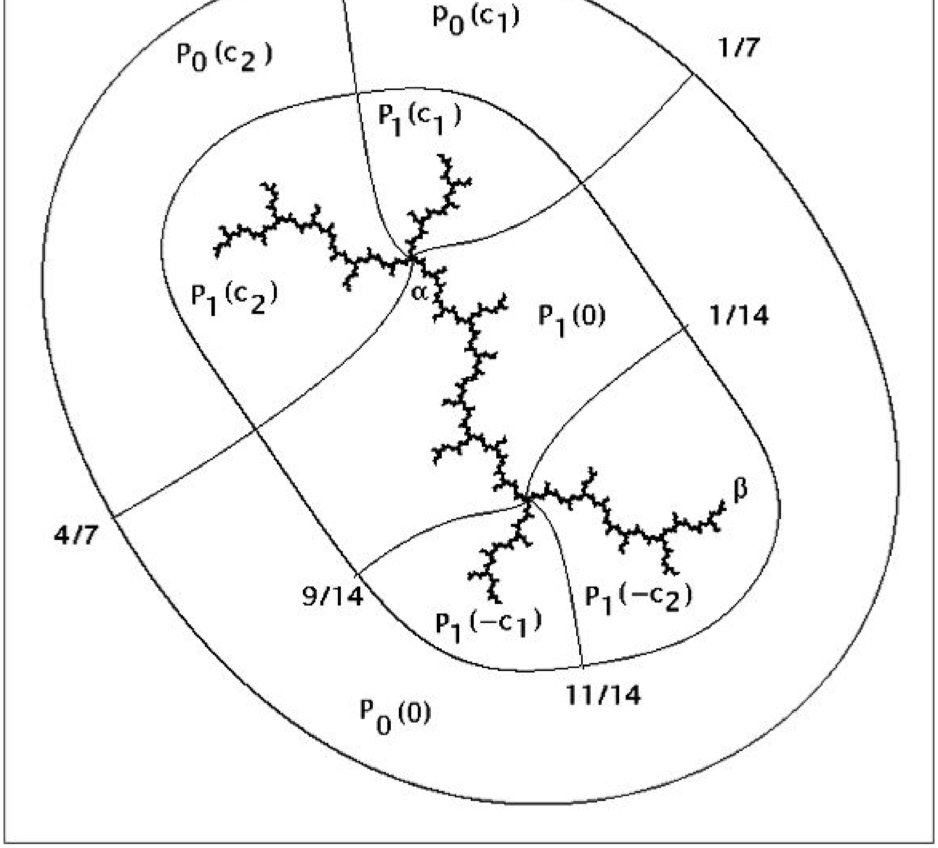
$$I = \cup I_i \text{ (disjoint union)}$$

so that $f^{n_i}\colon I_i \to I$ has bounded distortion. In some cases we can, but in many cases this is hopeless: the only way to get $f^{n_i}\colon I_i \to I$ surjective would be to allow not only for unbounded distortion, but even for unbounded degree.

- The next best thing is to have a countable Markov description: to construct a partition I_i so that $f^{n_i}|I_i$ maps onto a union of partition elements I_j. One can view this as a random walk.

- Here, and in general in holomorphic dynamics, we use essentially this approach: We take a sequence of partitions P_n of the complex plane with the property that some iterate of f maps elements of the partition P_n onto a union of elements of P_{n-1}.





The boundaries of the puzzles consist of special curves and there is a 'natural' parametrization on their boundary. (Any polynomial f of degree d is conformally conjugate to $z \mapsto z^d$ near ∞. W.r.t. these coordinates the rays correspond to straight lines, and the equipotentials to circles. In particular, f and $\tilde{f}$ are conformally conjugate near

∞ and this conjugacy is natural w.r.t. the rays and equipotentials.)

The main technical hurdle in our paper is to obtain control on the shape of the 'puzzle-pieces' of this Markov map. That is enough, because of a new way in which we are able to construct quasiconformal conjugacies:

Theorem 8 (QC-Criterion). *For any constant $\epsilon > 0$ there exists a constant K with the following properties. Let $\phi\colon \Omega \to \tilde{\Omega}$ be a homeomorphism between two Jordan domains. Let X be a subset of Ω consisting of pairwise disjoint topological open discs X_i. Assume moreover,*

1. *for each i both X_i and $\phi(X_i)$ have ϵ-bounded geometry and moreover*

$$mod(\Omega - X_i), mod(\tilde{\Omega} - \phi(X_i)) \geq \epsilon$$

2. *ϕ is conformal on $\Omega - X_i$.*

Then there exists a K-qc map $\psi\colon \Omega \to \tilde{\Omega}$ which agrees with Ω on the boundary of Ω.

2.4 The strategy of the proof of QC-rigidity

So the proof of the rigidity theorem relies on the following steps:

1. Associate to the polynomial f a suitable sequence of partitions $\mathcal{P}_n$.
2. Let Ω_n be a union of puzzle piece containing the critical points, and show that one has control of the domains of the first return map to Ω_n, as in the previous criterion. This is only true provided one constructs the sequence of partitions $\mathcal{P}_n$ very carefully.
3. Do the same for the topologically conjugate polynomial $\tilde{f}$.
4. Because of the above properties and the QC-criterion there exists a qc homeomorphism $h_n\colon \Omega_n \to \tilde{\Omega}_n$ which preserves the natural parametrization on the boundary;
5. Because $h_n\colon \Omega_n \to \tilde{\Omega}_n$ is natural on the boundary, the above qc map h_n can be extended to a global homeomorphism h_n

which is quasi-conjugacy and so that

$$h_n \circ f(x) = \tilde{f} \circ h_n(x)$$

for each $x \notin \Omega_n$.

6. Since K-qc homeomorphisms form a compact space, we can extract a K-qc limit of the sequence h_n. As Ω_n shrinks to a point, the limit h is a conjugacy.

By far the hardest part is to prove moduli estimates. Our original proofs can now be simplified using a lemma of Lyubich and Kahn, and can be applied to complex maps which are not infinitely renormalizable.

3 Maps which are typical in the sense of Lebesgue

3.1 Maps with a randomly chosen parameter are not hyperbolic

The set of hyperbolic polynomials does not have full measure with the space of all polynomials. This is a consequence of a theorem of Jakobson from the early 1980's. This theorem states that

Theorem 9. *The set of parameters a for which $f_a(x) = ax(1-x)$ has an absolutely continuous invariant (probability) measure μ has positive Lebesgue measure.*

So for a randomly chosen parameter a the orbits of $f_a(x) = ax(1-x)$ do NOT go to a periodic orbit. If there exists the invariant measure as in the theorem, one still can make statistical statements: For example the frequency of times iterates visit some open interval U is equal to $\mu(U)$. Even better, for any smooth function $\phi\colon \mathbb{R} \to \mathbb{R}$

one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) \;=\; \int \phi \, d\mu$$

for almost all x. So in this case, even if one cannot forecast what happens long ahead of time, one can give predictions about averages.

3.2 The really bad maps

In the one-dimensional case, the Palis conjecture states that most maps are 'not bad': they form a set of zero Lebesgue measure in the space of all polynomials.

Of course there are many bad maps. Hofbauer and Keller gave several examples of quadratic maps f which have various types of bad behaviour:

- for some bad maps, typical orbits stay most of the time near a repelling orbit: for a.e. x, $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ converges to a repelling orbit;

- for some other bad maps, for a.e. x, $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ does not converge in any sense.

This sort of bahaviour is bad. It means that no sort of long-term prediction is possible. Not even about averages.

3.3 Part II of the Palis conjecture. Is it typical for physical measures to exist?

One of the main challenges in the theory of dynamical systems is to solve the following:

Question: Is it true that C^k-generically, a diffeomorphism has at least one physical measure (and at most finitely many)?

Here we say that an f-invariant measure μ is *physical* or *SRB*, if the set $B(\mu)$ of points x such that for every continuous functions $\phi\colon \mathbb{R} \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) \;=\; \int \phi d\mu$$

has positive Lebesgue measure.

Definite progress was made towards this question if we are allowed to make C^1 perturbations, by a number of people. However in the C^k case, $k \geq 2$, no progress seems in sight. In the one-dimensional case there is much more progress as will see.

4 Existence of ACIM

Proving the existence of invariant goes back a long way.
The hyperbolic case. One of the classical cases one considers is when there is a partition I_i of an interval I

$$I = \cup I_i \text{ (disjoint union)}$$

so that $g\colon I_i \to I$ is a C^2 diffeomorphism which is expanding $|g'(x)| > \lambda > 1$. More precisely one asks

$$\log|Dg(x)| - \log|Dg(y)| \le K|f(x) - f(y)|.$$

Combining this with the the expansion gives that all iterates of g have bounded distortion: there exists K_1 so that

$$(BDI) \qquad \log|Dg^n(x)| - \log|Dg^n(y)| \le K_1$$

for x, y in some domain of continuity of g^n. For proving that there is an absolutely continuous invariant measure it is enough to have that for each $\epsilon > 0$ there exists $\delta > 0$ so that if A is a Borel set with $\lambda(A) < \epsilon$ then $\lambda(f^{-n}(A)) < \delta$ for all $n \ge 0$. This follows immediately from (BDI). In fact one can prove more: there exists the density has bounded variation. So one can define a Perron-Frobenius operator $P\colon E \to E$ on a suitable Banach space of functions (i.e. densities of measures) in such a way that $P(\tau)$ is the density of the push-forward

$$g_*(\tau\, d\lambda)$$

of the measure $\tau\, d\lambda$. This operator will have an eigenvalue equal to one (corresponding to the density of an invariant measure) and - if all is well - the rest of the spectrum is well-inside the unit circle.

Difficulties in the non-hyperbolic case. For more general maps there are three problems:

1. images of branches can become short
2. the critical points means that if there is a density for an invariant measure, it will definitely have poles; so what space to work in?
3. even the map has unbounded distortion; so to control the non-linearity of iterates of branches is even more difficult.

Some prior results. Misiurewicz proved that an acip exists for an S-multimodal map without periodic attractors or recurrent critical points. Collet-Eckman proved that an S-unimodal map f satisfying the following condition (the *Collet-Eckmann* condition) has acip:

$$(CE) \qquad \liminf_{n\to\infty} \frac{\log |(f^n)'(f(c))|}{n} > 0,$$

where c denotes the critical point of f. Nowicki and I improved this by showing that the following summability condition guarantees the existence of an acip for an S-unimodal map:

$$(NS) \qquad \sum_{n=0}^{\infty} \frac{1}{|(f^n)'(f(c))|^{1/\ell}} < \infty,$$

where ℓ is the order of the critical point c. Moreover, it was proved that the density of the acip with respect to the Lebesgue measure belongs to L^p for all $p < \ell/(\ell-1)$. (Note that this regularity is the best possible since the density is never $L^{\ell/(\ell-1)}$.)

Much more recently, Bruin, Shen and myself proved that no growth condition is needed at all:

Theorem 10. *In fact there exists constant $C(f)$ such that if*

$$\liminf_n |(f^n)'(f(c))| \geq C$$

for each critical point c then f has an acip.

4.1 Strategy of proof of the theorem on the existence of acim's without growth conditions

We say that f satisfies *the backward contracting property with constant* r ($BC(r)$ in short) if the following holds: there exists $\epsilon_0 > 0$ such that for each $\epsilon < \epsilon_0$, any critical points $c, c' \in \mathrm{Crit}(f)$ and any component W of $f^{-s}(B_{r\epsilon}(f(c')))$, $s \geq 1$

$$W \cap B_\epsilon(f(c)) \neq \emptyset \text{ implies } |W| \leq \epsilon. \tag{1}$$

We say that f satisfies $BC(\infty)$ if it satisfies $BC(r)$ for all $r > 1$. Clearly, for any $r > 1$ the property $BC(r)$ implies that f has no critical relation, i.e., no critical point is mapped into the critical set under forward iteration.

The proof of Theorem 10 breaks into the following two propositions.

Proposition 1. *If*

$$\liminf_n |(f^n)'(f(c))| \geq C$$

for each critical point c then f satisfies property $BC(r)$ where r depends on C.

Proposition 2. *There exists $r(f)$ such that if f satisfies the $BC(r)$ then there exists a constant M such that for every Borel set A we have*

$$|f^{-n}(A)| \leq M|fA|^{\kappa/\ell_{\max}}. \tag{2}$$

It follows from the inequality (2) that any weak limit μ of

$$\frac{1}{n}\sum_{i=0}^{n-1}(f^i)_* Leb$$

is an acip of f. In fact the density of μ with respect to the Lebesgue measure is in L^p, where $p \to \ell_{\max}/(\ell_{\max} - 1)$ as $\kappa \to 1$. So the previous propositions imply the theorem.

The proof of Proposition 1 is not hard: it is based on the one-sided Koebe principle:

Proposition 3. *Let $h : [a, b] \to \mathbb{R}$ be a C^3 diffeomorphism onto its image with negative Schwarzian. Then the following holds:*

1. *(the Minimum Principle) $|h'|$ does not take its minimum in (a, b).*

2. *(the One-sided Koebe Principle) Let $x \in (a, b)$ be such that $|h(a) - h(x)| \geq \tau |h(x) - h(b)|$. Then $|h'(x)| \geq C(\tau)|h'(b)|$, where $C(\tau)$ is a constant.*

3. *(the Koebe Principle) Let $a < x < y < b$ be such that*

   $$|h(x)-h(a)| \geq \tau |h(y)-h(x)| \text{ and } |h(b)-h(y)| \geq \tau |h(y)-h(x)|.$$

   Then
   $$\left(\frac{\tau}{1+\tau}\right)^2 \leq \frac{|h'(x)|}{|h'(y)|} \leq \left(\frac{1+\tau}{\tau}\right)^2.$$

Note: Negative Schwarzian

$$Sf = f'''/f' - (3/2)(f''/f')^2 < 0$$

looks a strong assumption. It is NOT. One can prove that any map which has no neutral periodic orbits is smoothly conjugate to one which has negative Schwarzian (modulo some mild other smoothness assumptions.)

4.2 The proof of Proposition 2

The proof Proposition 2 goes in several steps:

Step 1: An open set $V \subset [0,1]$ is called *nice* if for each $x \in \partial V$ and for any $k \geq 1$, $f^k(x) \notin V$.

For $\lambda > 0$ we say that a nice open set V is λ-*nice* if for each return domain J of V, we have $(1+2\lambda)J \subset V$.

Lemma 1. *For any $f \in \mathcal{A}$ and each $\lambda > 1$ there exists $r > 2$ such that if f satisfying $BC(r)$ and any $\epsilon > 0$ sufficiently small, the following holds: for each $c \in Crit(f)$, there exists an open interval V_c such that $V := \bigcup_{c \in Crit(f)} V_c$ is nice and such that*

- *for each $c \in Crit(f)$ we have*

$$\hat{B}_\epsilon(c) \subset V_c \subset \hat{B}_{2\epsilon}(c);$$

- *V is λ-nice.*

Here $\hat{B}_\epsilon(c)$ is defined to be the connected component of $f^{-1}(B_\epsilon(f(c)))$ containing c.

Proof. The proof follows from the following argument due to Rivera-Letelier. For $\epsilon > 0$ small, define the open set $V^n = \bigcup_{i=0}^{n} f^{-i}(\hat{B}_\epsilon)$. Clearly V^∞ is nice. Take V_c^n to be the connected component of V^n which contains c, and let $V_c = V_c^\infty$. It remains to show that $V_c^n \subset \hat{B}_{2\epsilon}(c)$ for each n. We do this by induction. For $n = 0$ this holds by definition, so assume it holds for n. Consider $Z = f(V_c^{n+1}) \setminus B(f(c), \epsilon)$. For $z \in Z$, there exists $m(z) \in \{0, 1, \ldots, n\}$ and $c_0(z) \in \mathrm{Crit}(f)$ so that $f^{m(z)}(z) \in \hat{B}_\epsilon(c_0(z))$. Now choose $z_0 \in Z$ so that $m_0 = m(z_0)$ is minimal among $m(z)$ for points $z \in Z$ and let $\hat{c}_0 = c_0(z)$. Since $f^{m_0}(z) \in \hat{B}_\epsilon(\hat{c}_0)$, and since $f^{m_0}(Z) \subset V_c^n$, the induction hypothesis implies

$$f^{m_0}(Z) \subset \hat{B}_{2\epsilon}(\hat{c}_0).$$

Since f satisfies $BC(2)$ and Z has distance ϵ to $f(c)$, it follows that $|f(Z)| < \epsilon$ and $Z \subset \hat{B}_{2\epsilon}(c)$. This completes the induction step.

To prove that V is λ-nice one uses some real bounds. $\square$

The following notion will then be useful.

If I is an interval which contains a critical point c and J is a uni-critical pull back of I then we say that J is a *child* of I. So there exists $c' \in \mathrm{Crit}(f)$, $s \geq 0$ and an interval $\tilde{J} \ni f(c')$ such that

- f^s maps $\tilde{J}$ diffeomorphically onto I;
- J is the component of $f^{-1}(\tilde{J})$ which contains c.

Lemma 2. *Let $c, c' \in Crit(f)$, let $I \ni c$ be a λ-nice interval and let*

$$J_1 \supset J_2 \supset \cdots \supset J_m$$

be children of I which contain c'. Then

$$|f(J_i)| \leq \rho^{i-1}|f(J_1)|$$

holds for all i, where $\rho = \rho(\lambda) > 0$ is a constant, and $\rho \to 0$ as $\lambda \to \infty$.

Step 2: Assume that f satisfied $BC(r)$. Then by Lemma 1 there exists a λ-nice puzzle piece I between $\hat{B}_\epsilon$ and $\hat{B}_{2\epsilon}$. By Lemma 2 we know that the children of this set I shrink fast.

Set $r_1 = r^{1/\ell_{\max}}$. For each $n \geq 0$ and $\delta > 0$, define

$$L_n(\delta) = \sup\{|f^{-m}(A)| : 0 \leq m \leq n, A \subset \hat{B}_{\epsilon_0/r_1} \text{ is an interval}, |fA| \leq \delta\}.$$

Lemma 3. *Let I be a λ-nice interval such that $\hat{B}_\epsilon(c) \subset I \subset \hat{B}_{2\epsilon}(c)$, where $c \in Crit(f)$ and $\epsilon < \epsilon_0$, and let A be an interval such that*

$$A \subset \hat{B}_{\epsilon/2}(c) \text{ and } A \not\subseteq \hat{B}_{\epsilon/r_1}(c). \tag{3}$$

Then for all $n \geq 1$,

$$|f^{-n}(A)| \leq C\frac{|A|}{|I|}|f^{-n}(I)| + 2N \sum_{i=1}^{\infty} L_{n-1}(\rho^i |fA|), \tag{4}$$

The proof goes by splitting collection of all components J of $f^{-n}(I)$ into the class $\mathcal{J}_0$ of components J of $f^{-n}(I)$ such that $f^n : J \to I$ is a diffeomorphism and $\mathcal{J}_1$ of all other components of $f^{-n}(I)$.

By the Koebe principle, for all $J \in \mathcal{J}_0$, we have

$$\frac{|f^{-n}(A) \cap J|}{|J|} \leq C\frac{|A|}{|I|}, \tag{5}$$

where C is a constant depending only on $\ell_{\max}$.

For each $J \in \mathcal{J}_1$, there exist a unique $0 \leq n_1 = n_1(J) < n$ and a unique interval $J' \supset f^{n-n_1}(J)$ such that J' is a child of I and such that $f^{n-n_1}(J)$ contains a critical point. Let A' a component of $f^{-n_1}(A) \cap J'$ then by Koebe

$$\frac{|f(A')|}{|f(J')|} \leq C \frac{|A|}{|f^{n_1}(J')|} \leq C' \frac{|A|}{|I|} \leq C'(2r_1)^{\ell_c - 1} \frac{|f(A)|}{|f(I)|}$$

where the 2nd inequality holds because $f^{n_1}(J')$ contains a component of $I \setminus A$ and last inequality because of non-flatness of f and (3). Because J' is a child and using Lemma 2 the claim follows.

Step 3: Using some induction argument it then follows that for each interval A inside some small neighbourhood U of $\mathrm{Crit}(f)$,

$$|f^{-n}(A)| \leq K|A|^{\kappa/\ell_{\max}}.$$

To get this estimate for general sets one uses a combinatorial argument and the minimum principle: (the derivative is closest to the boundary of a branch).

5 Ubiquity of having a physical measure

In the one-dimensional setting the situation is better understood.

Theorem 11 (Existence of SRB measures). *For Lebesgue almost all parameters $c \in \mathbb{R}$, the map $\mathbb{R} \ni z \mapsto z^\ell + c$ has a unique SRB measure, which is either*

- *absolutely continuous or*
- *its support is equal to $\omega(0)$ (the omega-limit set of the critical point 0) and $f|_{\omega(0)}$ is uniquely ergodic.*

This was proved by Lyubich for the case that $\ell = 2$, and for ℓ any even integer this was proved by Bruin, Shen and SvS. In fact for $\ell = 2$, for typical parameters $c \in \mathbb{R}$ the SRB measure is absolutely continuous or concentrated on an attracting hyperbolic periodic orbit.

Our proof of the previous theorem works also for general one-dimensional maps, except in one place. Hopefully this is just a technical difficulty.....

5.1 Strategy of proof

We do not aim to get positive Lyapounov exponents. To get an acim, we are showing that for **all** parameters c

- f_c has low combinatorial complexity and (therefore) a unique physical measure, or
- f_c has the decay of geometry condition.

Then we show that for **almost all** parameters c which have the decay of geometry condition the following summability condition $\mathcal{SC}$ holds:

$$\sum_{n=0}^{\infty} \frac{1}{|Df_c^n(c)|^\alpha} < \infty \text{ for any } \alpha > 0. \tag{6}$$

As we have seen such parameters have an absolutely continuous invariant probability measure.

5.2 Reluctant versus persistent recurrence

Let us first define some parameter sets.

As mentioned, an interval $J \ni 0$ is a called a *child* of I if it is a unimodal pullback of I, i.e., if there exists an interval $\tilde{J}$ containing the critical value c and an integer $s \geq 0$ so that $f^{s-1} \colon \tilde{J} \to I$ is a homeomorphism and $J = f^{-1}(\tilde{J})$.

If f_c is non-renormalizable, recurrent and there exists a nice interval $I \ni 0$ with infinitely many children, then we say that f_c is *reluctantly recurrent*; otherwise it is called *persistently recurrent.*

Reluctantly recurrent means that you can go from arbitrarily small neighbourhoods of the critical value diffeomorphically to big scale. This is the easy case.

5.3 The Decaying Geometry Condition

Let us say that a parameter c for which the critical point of f_c is recurrent and f_c is non-renormalizable, has *decaying geometry property* if either

- f_c is reluctantly recurrent, or
- f_c is persistently recurrent and there exists a sequence of nice intervals $\Gamma^0 \supset \Gamma^1 \supset \cdots \ni 0$ such that for each $n \geq 0$, Γ^{n+1} is the smallest child of Γ^n, and so that $|\Gamma^{n+1}|/|\Gamma^n| \to 0$ as $n \to \infty$.

Let $\mathcal{DG}$ denote the collection of parameters c for which f_c satisfies the decaying geometry condition. We should note that if $\ell = 2$, **all** parameters have the decaying geometry property. This is definitely NOT true when $\ell > 2$.

5.4 Combinatorial classification

Without going into details, we define the notion of low combinatorial complexity and show

Theorem 12. *If f is persistently recurrent and has* low combinatorial complexity, *then $f|\omega(0)$ is uniquely ergodic. Moreover, if f is C^3, then there is a unique physical-measure which is either an acip, or is the unique invariant probability measure supported on $\omega(0)$.*

It follows from Lebesgue ergodicity of unimodal maps without periodic attractors that an acip is indeed the unique physical-measure. The link with the decaying geometry property is made in the following:

Theorem 13. *If $c \notin \mathcal{DG}$, then f_c has low combinatorial complexity (and hence Theorem 12 applies).*

So it is enough to consider only parameters in $\mathcal{DG}$.

5.5 Parameters in $\mathcal{DG}$

Pick a parameter $c \in \mathcal{DG}$. Then by definition either the critical point is reluctantly recurrent (and one can go to big scale) or it is persistently recurrent and there exists a sequence of nice intervals $\Gamma^0 \supset \Gamma^1 \supset \cdots \ni 0$ such that for each $n \geq 0$, Γ^{n+1} is the smallest child of Γ^n, and so that $|\Gamma^{n+1}|/|\Gamma^n| \to 0$ as $n \to \infty$.

From this one can show that there exists a critical puzzle piece V_c so that the first return map g_c to V_c

$$g_c \colon \cup_i U_{i,c} \to V_c$$

has the property that the moduli of $V_c \setminus U_{i,c}$ are as large as we want.

To state a sharper version of this, we first give some definitions. Given a topological disk $\Omega \subset \mathbb{C}$ and a set A, define

$$\lambda(A|\Omega) = \sup_{\varphi} \frac{m(\phi(A \cap \Omega))}{m(\phi(\Omega))},$$

where φ runs over all conformal maps from Ω into $\mathbb{C}$ and m denotes the planar Lebesgue measure.

Definition 1. Let $V \subset \mathbb{C}$ be a topological disk. Let U_i, $i = 0, 1, \ldots$ be pairwise disjoint topological disks contained in V. We say that the family $\{U_i\}$ is *ϵ-absolutely-small in V* if $\lambda(\bigcup_i U_i|V) < \epsilon$, and for each i, the diameter of U_i in the hyperbolic metric of V is less than ϵ. Here the hyperbolic metric of V is the pullback of the standard Poincaré metric on the unit disc $\mathbb{D} \subset \mathbb{C}$ by the Riemann mapping from V to $\mathbb{D}$.

Theorem 14. *Consider a map $f = f_c$ with $c \in \mathcal{DG}$. Then for any $\epsilon > 0$, there exists a critical puzzle piece Y such that the collection of the components of the domain of the first return map to Y is ϵ-absolutely-small in Y.*

5.6 Robustness under changes in the parameter

Now consider c_0 as above and the first return map

$$g_{c_0} : \bigcup_i U_{i,c_0} \to V_{c_0} := Y$$

Also consider the set of parameters $D \ni c_0$ so that the puzzle piece V_c (consisting of rays and equipotentials) still exists (moves holomorphically) for all $c \in D$ and so that the first return

$$\tilde{g}_c : \bigcup_i \tilde{U}_{i,c} \to \tilde{V}_c$$

is pseudo-conjuate to $g_{c_0} : \bigcup_i U_{i,c_0} \to V_{c_0}$.

Here a qc map $\phi : \mathbb{C} \to \mathbb{C}$ is called *a pseudo-conjugacy* between these maps if ϕ maps V onto $\tilde{V}$, U_i onto $\tilde{U}_i$, and respects the boundary dynamics: for each $z \in \partial U_i$, $\phi \circ g(z) = \tilde{g} \circ \phi(z)$.

Proposition 15. *Let g and $\tilde{g}$ be first return maps as above, and let ϕ be a pseudo-conjugacy between them which is conformal a.e. in $V \setminus (\bigcup U_i)$. There exists a universal constant $\epsilon_0 > 0$ such that provided that $\{U_i\}$ is ϵ_0-absolutely-small in V, there exist*

- *a 2-qc map $\chi_0 : V \to \tilde{V}$ such that $\chi_0 = \phi$ on ∂V;*
- *a qc pseudo-conjugacy map $\psi : \mathbb{C} \to \mathbb{C}$ such that $\psi = \phi$ on $\mathbb{C} \setminus \bigcup_i U_i$ such that ψ is 2-qc on $V \setminus \overline{U_0}$.*

So the information on

$$g_{c_0} : \bigcup_i U_{i,c_0} \to V_{c_0} := Y$$

holds also for

$$g_c : \bigcup_i U_{i,c} \to V_c$$

for all $c \in D$. As $c \in D$ goes to the boundary the domain $U_{0,c}$ containing the critical point degenerates. That is why ψ is 2-qc only on $V \setminus \overline{U_0}$.

5.7 Decay of Geometry and the $\mathcal{SC}$ condition

To get that the $\mathcal{SC}$ condition holds for almost all parameters in $\mathcal{DG}$, we use the following lemma:

Lemma 4. *Let $g : \bigcup U_i \to V$ be a polynomial-like map such that U_0 contains the critical point of g. Let W be a domain of the first return map to U_0 (under g) with return time s. Then*

$$mod(U_0 \setminus W) \geq \frac{1}{\ell}((s-1)mod'(g) + mod(g)) \tag{7}$$

where we define $mod(g) = mod(V \setminus \overline{U_0})$ and $mod'(g) = \inf_i mod(V \setminus \overline{U}_i)$.

So provided the first return map visits sufficiently many non-central domains, the modulus grows.

So to get maps which 'most maps' are in $\mathcal{SC}$ it is enough to show that maps so that the critical point only visits a bounded number of central domains at each step, are rare.

5.8 Going from Dynamical to Parameter Space

Assume D, U, V are Jordan discs in $\mathbb{C}$. Let $h_c\colon V \to \mathbb{C}$, $c \in D$, be a holomorphic motion, i.e., $z \mapsto h_c(z)$ injective, for any $z \in V$ the map $D \ni c \mapsto h_c$ holomorphic and $h_0 = id$.

$\phi\colon D \to \mathbb{C}$ is called a *diagonal* of h if $\phi(c) \in h_c(V)$, ϕ extends to $\overline{V}$ and $c \mapsto h_\lambda^{-1} \circ \phi(c)$ is a homeomorphism from ∂D to ∂V. (By the argument principle, this implies that $h_c(z) = \phi(c)$ has a unique solution.)

Lemma 5. *There exists $M > 0$ with the following property. Assume $mod(V \setminus \overline{U}) > 3M$. Assume that for each $c \in D$, h_c is* controlled*: there exists a 3-qc map $\hat{h}_c : \mathbb{C} \setminus \overline{U} \to h_c(\mathbb{C} \setminus \overline{U})$ which coincides with h_c on $\partial V \cup \partial U$.*

Then $D' = \{c \in D \mid h_c^{-1}(\phi(c)) \in U\}$ is a topological disk, and

$$mod(D \setminus \overline{D'}) \geq \frac{1}{3} mod(V \setminus \overline{U}) - M.$$

5.9 Parameter removal

We start with a map g_c in $\mathcal{DG}$. From this it follows that if we consider the first return map g_c, $c \in D$ to appropriate critical puzzle pieces $P_n(c)$

$$\mathbf{g} = \{g_c | \cup_i U_{i,c} \to V_c, c \in D\}$$

then the modulus (in the sense as before) is as large as we want.

It is easy to show that there is a holomorphic motion of these puzzle pieces and that $c \mapsto g_c(0)$ forms a diagonal.

Then we apply an inductive scheme where we remove the parameters which only visit a small number of domains.

- the polynomial-like parameters which we keep, will have exponentially growing moduli. This implies that $\mathcal{SC}$ holds.
- since the moduli grow, the proportion of the parameters which we remove also gets exponentially smaller.