The Value of Embedded Real Options: Evidence from Consumer Automobile Lease Contracts - A Note

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ABSTRACT

In a recent article, Giaccotto, Goldberg, and Hedge (2007) provide a simple model for pricing the cancellation and the purchase options typically embedded in automobile lease contracts, assuming constant interest rates. They show that the cancellation option is worthless because of a penalty applied if the lease is terminated before maturity. We develop a model with stochastic interest rates, to show that the cancellation option generally has a significant value also in presence of the penalty and provide sufficient conditions to make the cancellation option worthless also in a stochastic term structure setting. This extends Giaccotto, Goldberg, and Hedge (2007) results to a more general framework.

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Introduction

In an automobile lease contract a financial institution (lessor) buys a new vehicle from the manufacturer’s franchised automobile dealer and leases it to a person or a firm (lessee) in exchange of a series of lease payments.

The most common type of automobile lease contract embeds two options: an American put option giving the lessee the right to extinguish the contract before expiration (cancellation option); a European call option to buy the vehicle for a predetermined residual value at the final date, in place of returning the vehicle to the lessor (purchase option). An early exercise of the lease arrangement using the cancellation option provokes the loss of the purchase option. Consequently there is an interaction effect between the two options, which should be properly accounted for when valuing the contract and when the cancellation option is exercised. Giaccotto, Goldberg, and Hedge (2007) shows that the presence of a cancellation penalty makes the cancellation option worthless, thus simplifying the task of valuing the options embedded into the lease contract.

It’s widely accepted that the two main sources of uncertainty in a lease contract are the dynamics of vehicle depreciation and the fluctuations of interest rates. Yet, the most important valuation models for lease contracts proposed so far assume constant interest rates, focusing only on the movements of car market value.\(^1\)

We propose a general model based also on stochastic interest rates.\(^2\) We show that, with stochastic interest rates, the cancellation option may be valuable also when the penalty is considered and we provide evidence that the interaction effect between the two options can be significant. Moreover, we discuss the conditions that make the cancellation option worthless in our more general setting. Overall, we contribute to this literature by extending the results of previous models for valuing lease contracts.

The paper is organized as follows. In Section I we describe the simple model with constant interest rates by Giaccotto, Goldberg, and Hedge (2007), to introduce our notation and discuss some of the features of the general valuation problem. In Section II we present the general model based on stochastic interest rates, under the assumption that the incentive to cease the lease contract for the lessee is related only to the deviation of the market value of the vehicle from the contractual value. In Section III we provide a more realistic description of the incentive of the lessee to exercise the cancellation

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\(^1\)In their work, Giaccotto, Goldberg, and Hedge (2007) at p. 420 state:

“...whether ignoring interest rate uncertainty is completely satisfactory is an open question that we can not resolve here.”

\(^2\)Several authors in securities valuation and mortgage valuation literature stressed the importance of a two–factor model to evaluate a cancellation option under both interest rates and market price uncertainty. See for instance Kau, Keenan, Muller, and Epperson (1995), Titman and Torous (1989) and Deng, Quigley, and van Order (2000).
option. We show also a set of conditions to make the cancellation option worthless in a framework with stochastic interest rates. In Section IV we present a numerical implementation of our model, based on Vasicek (1977) model for interest rates, to show the actual value of the cancellation option and the interaction effect of the two options embedded in a lease contract. Section V summarizes our results.

I. The Giaccotto, Goldberg, and Hedge (2007) model

Schallheim and McConnel (1985) have shown that the cancellation option value can be a significant part of the overall contract value in absence of market frictions. Giaccotto, Goldberg, and Hedge (2007) document the impact of those real world market frictions on the cancellation option value. The most important friction emerging from their empirical study is a cancellation penalty clause embedded in almost all negotiated contracts. Following McConnel and Schallheim (1983), Giaccotto, Goldberg, and Hedge (2007) provide a valuation model for a $T$–month cancelable lease as a combination of a one–month lease value plus the value of a stream of $(T – 1)$ one–month call options to renew the lease at the subsequent dates. Consistently with the main strand of literature, Giaccotto, Goldberg, and Hedge (2007) focus on closed-end leases; namely, lease contracts such that the lessee is not responsible for any value shortfall with respect to the contractual residual value of the leased vehicle at maturity. In this section, for convenience and to introduce the more general framework used in the subsequent sections we summarize their model.

For simplicity we assume that the cancellation option can be exercised only at the lease payment dates. The set of payment dates is $\{0, 1, \ldots, T\}$, where $T$ denotes the contract expiration date, and the length of the unit time period is one month. Consistently with Giaccotto, Goldberg, and Hedge (2007), in this section we assume a non-stochastic and flat term structure of interest rates, with market rate $r$ on an monthly basis.

We denote $V_t$ the market value of the leased vehicle after $t$ months since the inception of the contract. We denote $X^s_t$ the contractual residual value of the leased vehicle at $t$, for a lease started at $s \leq t$. In general, $X^s_t$ should be the future (i.e., the certainty

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3Assuming constant interest rates and not considering any market friction, their assessment of the cancellation option value ranges from 2% to 32% of the underlying asset value across a range of realistic values of the parameters.

4See Miller (1995) for a discussion about closed–end and open–end leases. In this note, we will deal with closed–end lease only.

5In the more general case, which assumes the cancellation option can be exercised at any date and not just at the payment dates, the results are almost identical. Details are available from the authors on request.

6In what follows we will use rates on a monthly basis to keep our notation simple. We will drop this assumption in Section IV.
equivalent) value at \( t \) of the vehicle as of the information available at \( s \), \( X^s_t = \hat{V}^s_t \), with \( X_t^s = V_t \). Let \( L^s_t \) be the lease payment due at \( t \) for a lease contract signed at \( s \leq t \). We assume that the sequences of contractual residual values, \( X^s_t \), and the sequence of payments, \( L^s_t \), are set at \( s \) so that the ex ante equilibrium between the two parties of the lease hold at any payment date. This means that the contractual residual value at every payment date is equal to the outstanding debt. Hence, for a contract signed at \( s \leq t \), with a monthly rate \( i_s \), the amount \( L^s_t \) must be the present value at \( t \) of the (contractual) value reduction of the vehicle plus the interest on the asset value (i.e., on the outstanding debt) in the period between \( t \) and \( t + 1 \)

\[
L^s_t = e^{-is} (X^s_t - X^s_{t+1} + X^s_t(e^{is} - 1)) = X^s_t - X^s_{t+1}e^{-is}.
\]

This is simply an extension to a multiperiod setting of the equilibrium lease payment by Miller and Upton (1976).

The value at \( t \) at the constant (non-stochastic) rate \( r \) of the lease signed at \( s \leq t \) is

\[
W^s_t = \sum_{k=t}^{T-1} L^s_k e^{-r(k-t)} + X^s_T e^{-r(T-t)}
\]

\[
= X^s_t - \sum_{k=t+1}^{T} X^s_k (e^{-is} - e^{-r}) e^{-r(k-1-t)} = X^s_t - \sum_{k=t+1}^{T} X^s_k \gamma^s_k
\]

where we used (1) and

\[
\gamma^s_k = e^{-r(k-1-t)} (e^{-is} - e^{-r}) .
\]

The gross incentive of the lessee to early terminate the contract at \( t \) is given by the difference (if positive) between the value of the leasing signed at \( s = 0 \) and the value of a new leasing, considering the current market price of the vehicle, \( V_t \):

\[
\Pi^s_t = W^0_t - W^t_t = X^0_t - V_t + \sum_{k=t+1}^{T} (X^t_k - X^0_k) \gamma^0_k.
\]

According to Giaccotto, Goldberg, and Hedge (2007), a penalty must be paid by the lessee at the date the cancellation option is exercised. If the contract signed at \( s = 0 \) is

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7This assumption is not limiting the generality of the model and of our discussion. All the results we present can be obtained also assuming \( X^s_t = E_s[V_t] \) for \( s < t \), where \( E_s[\cdot] \) is expectation conditional on information available at \( s \). Notice that \( \hat{V}_t^s = E^*[V_t] \), where \( E^*[\cdot] \) denotes expectation under the risk-neutral probability.

8Notice that if \( i_s = r \), then \( W^s_t = X^s_t \).

9For the new contract, we are assuming the same contractual rate as before, \( i_t = i_0 \). This is not a limitation in a setting with constant and non-stochastic interest rates. In a subsequent section, in a more general framework, we will deal also with a possible different new contractual rate, \( i_t \neq i_0 \).
exercised at $0 < t < T$, the penalty is the difference (if positive) between the value of the lease minus the current value of the vehicle:

$$P_t = W_t^0 - V_t = X_t^0 - V_t - \sum_{k=t+1}^{T} X_k^0 \gamma_k.$$  \hfill (5)

The net incentive to terminate the lease is the gross incentive in (4) minus the penalty in (5)

$$\pi_t^c = \max \{ 0, \Pi_t^c - P_t \} = \max \left\{ 0, \sum_{k=t+1}^{T} X_k^t \gamma_k^0 \right\},$$  \hfill (6)

We note that the net payoff is a function of the market value of the vehicle at $t$ through the (new) residual values of the vehicle at the future dates, $\{X_{t+1}, \ldots, X_T\}$.

We denote $F$ the joint value of the cancellation and purchase options, accounting for their interaction.\footnote{Giaccotto, Goldberg, and Hedge (2007) define the penalty as $P = \max \{ 0, OLP - UIC - SUR \}$, where $OLP$ are the Outstanding Lease Payments, $UIC$ are the Unearned Interest Charges and $SUR$, or Surplus, is the market value of the usage rights, given by the difference between the market value of the vehicle and its contractual residual value.} The gain at $T$ from exercising the purchase option (if the cancellation option has not been exercised) is $\pi_T^p = \max \{ 0, V_T - X_T^0 \}$. The cancellation option can be exercised at any payment date before $T$. Therefore, $F_t$ is the solution of a Bellman equation such that $F_T = \pi_T^p$ at $T$, and at any $0 \leq t < T$

$$F_t = \max \left\{ \pi_t^c, e^{-r}E_t[F_{t+1}] \right\}.$$  \hfill (7)

In general, $i_0$ does not need to be equal to $r$. But if $i_0 = r$, it is easy to see from (3) that $\gamma_k = 0$ for $k = t+1, \ldots, T$, and so $\pi_t^c \equiv 0$. Hence, assuming that the contractual rate $i_0$ is equal to the market interest rate $r$ (as it is somehow natural in a framework with deterministic and flat term structure) make the cancellation option always worthless independently of the dynamics of the market value of the vehicle, and the only valuable option in the hands of the lessee is the purchase option. As a consequence, the joint value of the options in (7) simplifies to

$$F_t = e^{-r(T-t)}E_t[\pi_T^p].$$

This is essentially the argument used by Giaccotto, Goldberg, and Hedge (2007).

Using the same assumptions as in McConel and Schallheim (1983), $V_t$ follows a geometric Brownian motion (under the actual probability measure)

$$dV_t = \alpha V_t dt + \sigma_V V_t dZ_t \quad \text{with} \quad V_0 \neq 0,$$  \hfill (8)

\footnote{Geske (1979) introduced the valuation of compound options. The cancellation option and the purchase options are compound options, and their joint value is different from the sum of their individual values. This effect is due to the interaction between the two options, as studied by Trigeorgis (1993).}
where $\alpha$ is the contractual depreciation rate, $\sigma_V$ is the volatility of the market price of the vehicle, and $dZ$ is the increment of a standard Brownian motion. In this case, the price at $t \leq T$ of the options embedded in the lease contract is

$$F_t = V_t e^{-(r-\hat{\alpha})(T-t)} \mathcal{N}(h + \sigma_V \sqrt{T-t}) - X_T^0 e^{-r(T-t)} \mathcal{N}(h),$$

where

$$h = \frac{\log \left( \frac{V_t e^{\hat{\alpha}(T-t)}}{X_T^0} \right) - \frac{\sigma^2 V^2}{2} (T-t)}{\sigma \sqrt{T-t}},$$

$\mathcal{N}(\cdot)$ is the cumulative standard normal distribution, $\hat{\alpha} = \alpha - \phi$ is the drift under the risk–neutral probability, from continuous–time CAPM, and $\phi$ is risk premium.$^{12}$

On the other hand, if the contractual rate $i_0$ is different from $r$, the cancellation option has a positive value and the analytic approach proposed above for valuing the options embedded in the lease contracts is not feasible. In this case, we have to employ a numerical approach to find the joint value of the two embedded options, $F$, using (7) in a backward induction procedure. In Section IV we will provide the numerical solution for a specific example.

II. Stochastic term structure of interest rates

This section shows how to incorporate a stochastic term structure model into the framework described above.$^{13}$ From equation (4), the gross payoff from the cancellation of the initial contract, under the assumption of stochastic interest rates, is the one in (4), with the different specification for $\gamma$:

$$\gamma^s_k = B(t, k - 1) \left( e^{-i_s} - B(k - 1, k) \right),$$

for $s \leq t$, where $B(t, k)$ denotes the price at $t$, of a zero coupon bond ensuring the payment of $\$1$ at $k \geq t$.

The main difference with respect to the setting in Section I is that now we have at least two ways to determine the penalty: we can evaluate the outstanding lease payments either using market interest rates, or at the contractual rate. We will consider both cases separately, starting from the one based on the market rates.

$^{12}$Both $\alpha$ and $\sigma_V$, and the risk premium $\phi$, are on a monthly basis, to be consistent with the rates of the analysis in this section. The factor $e^{-(r-\hat{\alpha})}$ is the continuous time equivalent for $\lambda$ as in Giacotto, Goldberg, and Hedge (2007), following Rubinstein (1976) and Geske (1979). Notice that, under the assumption that the contractual residual value of the vehicle is the future price, i.e. $X_T^0 = V_T e^{\hat{\alpha}(T-t)}$, equation (9) reduces to $F_t = e^{-r(T-t)} \left( X_T^0 \mathcal{N}(h + \sigma_V \sqrt{T-t}) - X_T^0 \mathcal{N}(h) \right)$, which is reminiscent of Black (1976) valuation formula for options on futures.

$^{13}$We will not need to specify the term structure model until Section IV, where we will provide our numerical analysis of the lease contract.
In the first case, the penalty in (5) is

\[ P_t = X_t^0 - V_t - \sum_{k=t+1}^{T} X_k^0 \gamma_k^0 \]  \hspace{1cm} (11)\]

and the net payoff of the cancellation option is

\[ \pi_t^c = \max \{ 0, \Pi_t^c - P_t \} = \max \left\{ 0, \sum_{k=t+1}^{T} X_k^t \gamma_k^0 \right\} . \]  \hspace{1cm} (12)\]

where the \( \gamma \)'s are specified in (10). To get the conclusion that the cancellation option is worthless, as in Giaccotto, Goldberg, and Hedge (2007) (and in Section I), we have to determine the conditions for \( \gamma_k^0 = 0, k = t + 1, \ldots, T \), in (12). It is easy to see from (10) that in a stochastic framework this cannot happen almost surely.\(^{14}\) Hence, the cancellation option is almost surely valuable in a stochastic term structure setting, with the benefit from cancellation defined as in (12).\(^{15}\)

In the second case, the penalty is determined using the contractual rate \( i_0 \), all \( \gamma_k^0 = 0 \) in (11) and it becomes \( P_t = X_t^0 - V_t \), if positive. Under this condition, the net payoff from cancellation becomes

\[ \pi_t^c = \max \left\{ 0, \sum_{k=t+1}^{T} \left( X_k^t - X_k^0 \right) \gamma_k^0 \right\} . \]  \hspace{1cm} (13)\]

To obtain that the cancellation payoff is worthless it is sufficient that either \( \gamma_k^0 = 0 \) or \( X_k^t = X_k^0 \), for all \( k = t + 1, \ldots, T \). As we said above, the first condition is almost impossible in a setting with stochastic interest rates. The second condition is true when \( V_t = X_t^0 \), under the assumption that the market price of the vehicle follows a geometric Brownian motion. This is a trivial case, because it happens just when the lessee is not interested to terminate the lease.

To summarize, we have shown that in a stochastic framework the cancellation option is generally valuable, no matter if we use the contractual or market rates to determine the cancellation penalty. This means that in this setting we have to compute the joint value of the cancellation and purchase options, \( F \), embedded in the lease contract using a numerical method to solve the Bellman equation

\[ F_t = \max \{ \pi_t^c, B(t, t + 1)E_t[F_{t+1}] \} \]  \hspace{1cm} (14)\]

recursively using a backward induction approach.

\(^{14}\)To put it differently, this can happen only if the term structure is non-stochastic, flat and \( i_0 \) is equal to the market rate, \( r \).

\(^{15}\)As we will see in Section III, this is not the only way to specify the net benefit.
III. Accounting for better financial conditions

In Section II, following Giaccotto, Goldberg, and Hedge (2007), the incentive for the lessee to early terminate the contract at \( t \) was the difference (if positive) between the contractual value of the leasing and its fair value at the current market price of the vehicle. From this definition we concluded that in a setting with stochastic interest rates the cancellation option is valuable almost surely.

In this section we argue that the definition of the gross incentive given in (4) is not describing the actual incentive to the lessee to terminate the lease before maturity. Giaccotto, Goldberg, and Hedge (2007), p. 416, state that:

"The lessee has an incentive to cancel the lease primarily under the following two conditions: (a) The market value of the leased car depreciates more rapidly than the contractual rate of depreciation, and (b) the market rate of interest drops below the rate specified in the lease contract."

In (4) only point (a) has been accounted for. This was correct under the assumption of constant interest rates. However, in a stochastic term structure framework, the lessee could have a rational propensity for ceasing the contract not just because the market value of the vehicle is lower than the contractual one, but also because of a change in interest rates.\(^{16}\) This is to say that the specification of the incentive used so far misses an important component. In particular, to (4) we add the gain to cease (or the opportunity cost of not ceasing) the contract in order to open a new lease contract at better financial conditions. The gross payoff turns to

\[
\Pi_t^c = W_0^t - W_t^t = X_t^0 - V_t + \sum_{k=t+1}^{T} (X_k^t \gamma_k^t - X_k^0 \gamma_k^0),
\]

(15)

where \( \gamma_k^s \), for \( s = 0 \) and \( s = t \), is defined in (10) and where \( i_t \), the contractual rate at \( t \) for the new lease contract, is generally different from \( i_0 \).

The payoff of the cancellation option net of the penalty in (11), which is determined using the market rates, is

\[
\pi_t^c = \max \left\{ 0, \sum_{k=t+1}^{T} X_k^t \gamma_k^t \right\}.
\]

(16)

This expression is different from (12) because the \( \gamma \)'s depend on the contractual rate \( i_t \) instead of \( i_0 \).

\(^{16}\)This is a rather general effect on financial contracts. See also Ingersoll and Ross (1992).
We are now in the position to determine the conditions to extend, to the case with stochastic interest rates, the results by Giaccotto, Goldberg, and Hedge (2007) that the cancellation option is worthless. First, the net payoff (16) is identically worthless, when all $\gamma_k = 0$, for all $k = t + 1, \ldots, T$. As we said above, this is almost impossible in a stochastic framework. A second condition is that the contractual rate of the new lease, $i_t$, is the internal rate of return of the contract given the current market value of the vehicle and the current term structure. More precisely, $i_t$ must be the solution of equation

$$V_t = W'_t,$$

where $W'_t = X'_t - \sum_{k=t+1}^T X'_k \gamma'_k$ from (2) with the $\gamma$’s defined in (10). Since $X'_t = V_t$, from (17) we get

$$\sum_{k=t+1}^T X'_k \gamma'_k = 0,$$

that is, the net payoff in (16) vanishes and the cancellation option becomes identically worthless. Under these assumptions, the joint value of embedded options, $F$, can therefore still be valued using a generalized version of Black and Scholes formula in (9) with stochastic interest rates, $F_t = B(t,T)\mathbb{E}_t[\pi_T^P]$.

So far we have assumed that the penalty is determined using market rates. Instead, if the penalty is determined at the initial contractual rate $i_0$, then $P_t = X^0_t - V_t$. Under this assumption, given the gross payoff (15), the net incentive for the lessee to early terminate the contract is

$$\pi^c_t = \max\left\{0, -\sum_{k=t+1}^T X^0_k \gamma^0_k\right\}.$$  

From this expression some interesting remarks can be derived.

The first is that the net payoff is never zero even if $V_t = X^0_t$, because the lessee could have an incentive to early terminate just in order to profit from better interest rates conditions, as given by a different $i_t$ with respect to $i_0$.

The second remark is that if we assume that the contractual rate $i_t$ is the internal rate of the new lease, from (18), the net payoff simplifies to

$$\pi^c_t = \max\left\{0, -\sum_{k=t+1}^T X^0_k \gamma^0_k\right\}.$$  

Differently from the case with the penalty determined using market rates, the cancellation option is not worthless. Specifically the incentive is a function of the term structure dynamics but independent of the dynamic of market value of the vehicle, because this last component is completely offset by the penalty.
IV. The value of the embedded options

In this section we provide a numerical solution of the valuation problem introduced in Section III for a given specification of the term structure model and for a set of parameters, to show to what extent the joint value of the two most important options embedded in a lease affect the overall value of the contract. In particular, in Section III, we have shown that the cancellation option is valuable in case the penalty is determined at $i_0$, and the new contractual rate is the internal rate of return at the current market conditions. This produces a net incentive $\pi_c^t$ as in equation (20).

For this case, we want to show first of all how the valuation algorithm can be implemented.\footnote{This is an extension also of the valuation models, like the one by Schallheim and McConnel (1985), which evaluate the two embedded options separately.} The second goal is to show how valuable the cancellation option can be in a setting with stochastic interest rates. Moreover, we want to analyze the interaction effect of the two options embedded in the lease contract. Lastly, we want to provide the size of the potential error we would make assuming that the cancellation option is worthless under these broader circumstances.

The results presented in this section are based on Vasicek (1977) term structure model model\footnote{This is done for simplicity and other more general term structure models can be used. For instance, in unreported results we implemented a valuation algorithm based on Hull and White (1990) model, confirming the results we present below. This alternative implementation is available from the authors on request.} with $r_0 \neq 0$,

$$dr_t = \beta(\theta - r_t)dt + \sigma_r dW_t \quad \text{with } r_0 \neq 0,$$

where $\beta$ is the mean reversion coefficient, $\theta$ is the long–term interest rate, and $\sigma_r$ is the volatility, and $dW$ is the increment of a standard Brownian motion. We assume that the market value of the vehicle follows the dynamics in (8), where $\alpha$ and $\sigma_V$ are considered on an annual basis, for the sake of this section.\footnote{At this point it is clear that we could have chosen a more realistic model for $V$. The choice of a geometric Brownian motion was motivated only by the possibility to have a closed–form valuation formula. However, since such a process may give $V_t > V_0$ with positive probability, it is unlikely that it can properly represent the dynamic of the price of a used vehicle.}

Moreover, $dZdW = \rho dt$, where $\rho$ is the correlation between the market value and the interest rate dynamics.

The valuation problem must be solved by a recursive application of the Bellman equation (14). We employ a numerical method to compute the continuation value, $B_t = \mathbb{E}_t[F_{t+1}]$. We use a version of Monte Carlo simulation, the Least Square Monte Carlo (LSM) method proposed by Longstaff and Schwartz (2001), which is suited to deal with the early exercise feature of the cancellation option. The numerical solutions have been obtained using 10,000 paths for $(V_t, r_t)$ and 500 time steps. The continuation value was approximated using a complete set of power polynomials with maximum order
\[ N = 3 \] for each \( r_t \) and \( V_t \). In any case, the cancellation option can be exercised only at the specified lease payment dates.

As a base case example, we consider a leasing contract with the following parameters: \( V_0 = 20,000 \) is the current value of the vehicle; the duration of the contract is \( T = 2 \) years; the expected depreciation rate is \( \alpha = -20\% \) per year; \( \sigma_V = 5\% \) is the annual volatility of the value of the vehicle; \( \phi = 5\% \) is the suited risk premium; \( r_0 = 5\% \) is the current short rate on an annual basis; the mean-reversion speed in the Vasicek model is \( \beta = 0.1 \); \( \theta = 6\% \) is the long–term value for the short rate; \( \sigma_r = 1\% \) is the volatility of the short rate; \( \rho = 0.5 \) is the correlation between the short rate and the market value of the vehicle.

Our choice of parameters (\( \theta > r_0 \)) entails that the rate of returns are likely to increase. So, in principle, the value of the cancellation option should not be overstated. Instead, in a period of decreasing rates (\( \theta < r_0 \)), its value is relatively larger because of the potential more favorable financial conditions of the alternative lease contracts. However, a sensitivity analysis of the value on these parameters has been conducted to make sure our results are not driven by this particular choice of parameters.

In all simulations, \( i_0 \) is exogenous and equal to 6\% (i.e., the long term rate), and \( i_t \) is the internal rate, defined as the solution of equation (17). We account for the possibility that the lessor charge a spread over \( i_t \). In our first analysis, we study the sensitivity of the values of the options on the spread over \( i_t \). In principle, the larger the spread and the lower the incentive to terminate the current lease. Hence, we should observe a negative effect on the value of the cancellation option. We assume that the spread can range from 0\% to 1\%.

We compute the stand–alone value of the cancellation option, \( c \), and of the purchase option, \( p \), together with their joint value, \( F \), which accounts for their interaction. We compute also the incremental value of the purchase option, \( \hat{p} \), given by the difference between \( F \) and \( c \). This is done to compare it to \( p \) and measure the impact of the interaction effect; i.e. the mistake we make if we value the two options separately instead of jointly. The joint value of the two options is related also to the initial vehicle price to assess the importance of these options in the overall contract value. To study the impact of the cancellation penalty introduced by Giaccotto, Goldberg, and Hedge (2007), we compute also the value of the cancellation option, \( c^* \), and the joint value of the two options, \( F^* \), assuming that no penalty is applied. The ratio of \( F \) over \( F^* \) describes the size of the impact of the penalty in a setting with stochastic interest rates.

The results in Table I show that the value of the cancellation option can be significant and comparable to the stand–alone value of the purchase option.\(^{20}\) Specifically we notice that, as the spread increases, its value decreases from about $400 to $340 and the

\(^{20}\)Since the values are from Monte Carlo simulation, in parenthesis we report also the standard error of the sample estimates.
spread | $c$ | $p$ | $\hat{p}$ | $F$ | $F/V_0$ | $c^*$ | $F^*$ | $F/F^*$
---|---|---|---|---|---|---|---|---
0% | 258.06 | 310.89 | 143.46 | 401.52 | 2.01% | 507.68 | 777.79 | 51.62%
 | (0.75) | (4.72) | (4.01) | | | | | |
0.25% | 191.31 | 310.89 | 188.11 | 379.42 | 1.90% | 357.75 | 662.09 | 57.31%
 | (0.86) | (4.72) | (4.17) | | | | | |
0.50% | 134.64 | 310.89 | 227.74 | 362.38 | 1.81% | 302.96 | 611.93 | 59.22%
 | (0.92) | (4.72) | (4.31) | | | | | |
0.75% | 94.15 | 310.89 | 255.08 | 349.23 | 1.75% | 286.79 | 596.04 | 58.59%
 | (0.86) | (4.72) | (4.42) | | | | | |
1.00% | 64.63 | 310.89 | 275.24 | 339.87 | 1.70% | 274.51 | 584.23 | 58.17%
 | (0.76) | (4.72) | (4.52) | | | | | |

Table I: The impact of the interest rate spread. $c$ is the stand–alone value of the cancellation option; $p$ is the stand–alone value of the purchase option; $F$ is the joint value of the two options, considering their interaction; $\hat{p}$ is the difference between $F$ and $c$. $c^*$ and $F^*$ are the value of the cancellation option and the joint value of the two options, respectively, assuming that no penalty is applied. These values are obtained using LSM method with 10,000 paths, 500 time steps, and approximation of the continuation value using a complete set of power polynomials with maximum order $N = 3$. In parenthesis we report the standard deviation of the sample estimate. Model parameters are: $V_0 = $20,000, $T = 2$ years; $\alpha = -20\%$; $\sigma_V = 5\%$, $\phi = 5\%$, $r_0 = 5\%$, $\beta = 0.1$, $\theta = 6\%$, $\sigma_r = 1\%$, and $\rho = 0.5$.

The joint effect of the two options ranges from about 2% to 1.7% of the initial price of the vehicle. The value of the stand–alone purchase option, $p$, is independent of the spread, because this option does not depend of the contractual rates of alternative leases. On the other hand, the incremental value of the purchase option, $\hat{p} = F - c$ does depend of the spread. In particular, since the spread has a significant negative effect on the cancellation option, while only slightly decreasing the joint value of the two options, $\hat{p}$ is an increasing function of the spread. Moreover, $\hat{p}$ is significantly lower than $p$, especially when there is no spread. This interaction effect is due to the fact the $\hat{p}$ is the value of the purchase option conditional on the cancellation option not being exercised. Of course, the likelihood of an early termination of the current lease is increased if the potential new contracts are more favorable. The difference between $p$ and $\hat{p}$ can be used to measure the valuation error we would make omitting this interaction. The impact of the penalty on the stand–alone value of the cancellation option and on the joint value of the two options, $F$, is significant. On average, it reduces the value of the options by about 55%, for our choice of parameters, although in no way the penalty make the cancellation option worthless in a setting with stochastic interest rates.

In Table II, we test the sensitivity of the option value to the duration of the lease contract. In particular, we assume that $T$ ranges from 1 to 3 years, which is a span including most of the actual leases. By visual inspection, we notice that also in this case the value of the cancellation option is significant. It ranges from about $330 for a
Table II: The effect of longer maturity. $c$ is the stand–alone value of the cancellation option; $p$ is the stand–alone value of the purchase option; $F$ is the joint value of the two options, considering their interaction; $\hat{p}$, is the difference between $F$ and $c$. These values are obtained using LSM method with 10,000 paths, 500 time steps, and approximation of the continuation value using a complete set of power polynomials with maximum order $N = 3$. In parenthesis we report the standard deviation of the sample estimate. Model parameters are: $V_0 = $20,000; $\alpha = -20\%$; $\sigma_V = 5\%$, $\phi = 5\%$, $r_0 = 5\%$, $\beta = 0.1$, $\theta = 6\%$, $\sigma_r = 1\%$, and $\rho = 0.5$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$c$</th>
<th>$p$</th>
<th>$\hat{p}$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>146.82</td>
<td>297.09</td>
<td>182.70</td>
<td>329.52</td>
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<tr>
<td></td>
<td>(0.43)</td>
<td>(4.46)</td>
<td>(4.11)</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>207.18</td>
<td>313.02</td>
<td>165.47</td>
<td>372.65</td>
</tr>
<tr>
<td></td>
<td>(0.61)</td>
<td>(4.73)</td>
<td>(4.18)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>258.06</td>
<td>310.89</td>
<td>143.46</td>
<td>401.52</td>
</tr>
<tr>
<td></td>
<td>(0.75)</td>
<td>(4.72)</td>
<td>(4.01)</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>299.29</td>
<td>298.90</td>
<td>122.51</td>
<td>421.80</td>
</tr>
<tr>
<td></td>
<td>(0.87)</td>
<td>(4.56)</td>
<td>(3.73)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>332.92</td>
<td>281.53</td>
<td>106.69</td>
<td>439.61</td>
</tr>
<tr>
<td></td>
<td>(0.99)</td>
<td>(4.31)</td>
<td>(3.46)</td>
<td></td>
</tr>
</tbody>
</table>

one–year contract to more than $440 for longer leases. The same effect is apparent also for $c$. This is in line with intuition, as far as the cancellation option value is concerned: the longer the life of the contract, the higher the probability that $V_t$ and $r_t$ fall in a state of nature favorable to the exercise of the cancellation option, and so the more valuable is the cancellation option. On the other hand, the stand–alone value of the purchase option, $p$, is a concave function of $T$. This is due to the fact that both the market price (the underlying asset value of the call option, $V_T$) and the contractual residual value (the strike price of the call option, $X_0^T$) of the vehicle are functions of $T$. For shorter maturities, the effect on the strike price dominates ($X_0^T = V_0e^{\hat{\alpha}T}$, with $\hat{\alpha} < 0$). For longer maturities the effect on $V_T$ dominates, so that the probability that the purchase option is out of the money is higher. It is worth noticing that the impact of $T$ on $c$ dominates the impact on $p$, so that $F$ is monotonic increasing. In any case, since this impact is larger on $c$, the incremental value of the purchase option, $\hat{p}$ is decreasing with respect to maturity.

In Figure 1 we show the impact of the correlation between the market value of the vehicle and the interest rates on the value of the options embedded in the lease contract, considering the cancellation penalty. We plot the values of the stand–alone cancellation and purchase options, $c$ and $p$ respectively, the incremental value of the purchase option, $\hat{p}$, and the joint value of the options, $F$. The individual option values are independent of correlation. This is because the purchase option is a function only of the market price of the leased vehicle, and the cancellation option is a function only of the interest rate
**Figure 1: The correlation effect.** These values are obtained using LSM method with 10,000 paths, 500 time steps, and approximation of the continuation value using a complete set of power polynomials with maximum order $N = 3$. In parenthesis we report the standard deviation of the sample estimate. Model parameters are: $V_0 = $20,000, $T = 2$ years; $\alpha = -20\%$; $\sigma_V = 5\%$, $\phi = 5\%$, $r_0 = 5\%$, $\beta = 0.1$, $\theta = 6\%$, $\sigma_r = 1\%$.

dynamic (see equation (20)). However, the joint value of the two options, considering also their interaction, is an increasing function of $\rho$. The explanation is rather simple. The value of the cancellation option increases if the interest rates decrease, because the lessee has the opportunity to exploit better financial conditions. On the other hand, the purchase option value is an increasing function of $V_t$. Hence, if $V_t$ decreases, $p$ is reduced. In case of a positive correlation coefficient, $\rho$, there is an offsetting effect between the two values, reducing the overall risk and increasing the value. In case of a negative $\rho$, the value of the two options are more likely to jointly move in the same direction, thus increasing the overall risk and reducing $F$.

**V. Conclusions**

We extend the model by Giaccotto, Goldberg, and Hedge (2007) to evaluate the option embedded in an automobile lease contract in presence of a penalty applied to the lessee if the contract is terminated before maturity, to a setting with stochastic term structure.
We provide conditions that extend to our more general framework their results that the cancellation option is worthless. Moreover, in a setting where those conditions are not valid and the cancellation option is valuable, we show the actual impact of the penalty on the joint value of the purchase option and cancellation options typically embedded in automobile lease contracts.
References


