An Improved Binomial Lattice Method for Multi–Dimensional Options

first draft January, 2001
current version March, 2007
Forthcoming, Applied Mathematical Finance

Andrea Gamba
Department of Economics
University of Verona (Italy)

Lenos Trigeorgis
Department of Public and Business Administration - University of Cyprus (Cyprus)

Correspondence to:
Andrea Gamba
Department of Economics
University of Verona
Via Giardino Giusti, 2
37129 Verona (Italy)
tel. ++ 39 045 80 54 921
fax. ++ 39 045 80 54 935.

The authors are grateful to Matteo Tesser for computation assistance. We wish to thanks the participants of the 5th Annual International Conference on Real Options - UCLA, and of seminars at University of Cyprus and University of Pisa for their valuable comments. We are responsible for all remaining errors. The paper previously circulated under the title A log-transformed binomial lattice extension for multi-dimensional option problems.
An Improved Binomial Lattice Method for Multi–Dimensional Options

Abstract

We propose a binomial lattice approach for valuing options whose payoff depends on multiple state variables following correlated geometric Brownian processes. The proposed approach relies on two simple ideas: a log–transformation of the underlying processes, which is step by step consistent with the continuous–time diffusions, as proposed by Trigeorgis (1991), and a change of basis of the asset span, to transform asset prices into uncorrelated processes. We apply an additional transformation to approximate drift–less dynamics. Even if these features are simple and straightforward to implement, we show that they significantly improve the efficiency of the multi–dimensional binomial algorithm. We provide a thorough test of efficiency compared to most popular binomial and trinomial lattice approaches for multi–dimensional diffusions. Although the order of convergence is the same for all lattice approaches, the proposed method shows improved efficiency.

Keywords: Option pricing, binomial lattice, multi–dimensional diffusion.

JEL classification: G13
Introduction

Complex options or contingent claims dependent on multiple state variables are common in financial economics, both in financial as well as in real investment valuation problems.\(^1\) Closed-form solutions to price such multi-dimensional options are available only in a few special cases, so numerical methods must be generally employed. A number of approaches have been proposed to numerically tackle option valuation problems. Broadly, these can be divided into three main categories: numerical solutions to partial differential equations (pde) such as finite difference methods (first introduced in finance by Brennan and Schwartz (1977)) and finite elements; Monte Carlo simulation methods (first introduced by Boyle (1977)); and lattice methods, first proposed by Cox et al. (1979) (CRR in what follows). Among these, although for some approaches they are special cases of finite differences, lattice methods are generally considered to be simpler, more flexible and, if dimensionality is not too large, more efficient than other methods. In this paper we propose a binomial lattice extension to evaluate contingent claims whose payoff depends on multiple state variables that follow joint (correlated) geometric Brownian processes.

In the one-dimensional case, a number of variations of the CRR lattice approach have been proposed to approximate the price of options on asset values following a geometric Brownian motion. Rendleman and Bartter (1979), Jarrow and Rudd (1983) and Hull and White (1988, footnote 4) for example, propose different choices of parameters for the up (and down) multiplicative steps and the (risk-neutral) probabilities. Trigeorgis (1991) proposed a log-transformed (LT) version of the binomial lattice approach. Boyle (1988) and Kamrad and Ritchken (1991) (KR in what follows) propose trinomial lattice methods whose accuracy depends on the choice of a “stretch parameter” that must be chosen up front. Improved accuracy can be achieved at the cost of increasing computational effort, so the efficiency needs to be assessed.\(^2\)

Moreover, several variations have been introduced to improve the effi-

---

\(^1\)On the financial side, option pricing models for contracts on several underlying asset values have been presented by Stulz (1982), Johnson (1987), Boyle (1988), and Boyle et al. (1989) regarding options on the maximum or the minimum of several asset values and by Hull and White (1987), Schwartz (1982) concerning problems with more than one state variable. In recent years, rainbow options have become popular financial instruments. Multi-factor real options have been studied by Hodder and Triantis (1990), Cortazar and Schwartz (1994), Geltner et al. (1995), Cortazar et al. (1998), Brekke and Schieldrop (2000), and others.

\(^2\)For a thorough comparison of lattice approaches in the one-dimensional case, see Broadie and Detemple (1996). They show that, for one-dimensional problems, trinomial methods are slightly more efficient than traditional binomial methods. In Section 3, we provide a comparison of Kamrad and Ritchken (1991) trinomial lattice approach with other methods in a multi-dimensional setting. The analysis shows that generally trinomial lattices are dominated by other lattice methods as far as efficiency is concerned.
ciency of the lattice method in the one-dimensional case. For brevity, only a few of them can be mentioned here. Hull and White (1988) applied control variate technique to lattice methods. Geske and Johnson (1984) and Breen (1991) suggested to use Richardson extrapolation to accelerate convergence. Broadie and Detemple (1996) introduce a modification of the binomial methods (named BBSR) that significantly outperform other lattice approaches in terms of efficiency. Leisen and Reimer (1996) and Leisen (1998) define an order of convergence for European and American plain vanilla options and provide a binomial lattice method with faster convergence and improved efficiency. Figlewski and Gao (1999) propose a further generalization of lattice methods with the property that the density of the tree is variable in order to provide higher accuracy in regions where the behavior of the underlying asset price is more relevant. Using the same argument, but in the opposite direction, Baule and Wilkens (2004) propose to properly prune the tree in regions with small probability for the underlying asset price.

Other lattice approaches have been proposed to cope with different stochastic processes (e.g. Nelson and Ramaswamy (1990)) or with time-varying variance-covariance structures (see Ho et al. (1995)) for the underlying asset values. A related but different branch of research is the one dealing with the complete markets property displayed by the CRR approach. Madam et al. (1989), He (1990) and Chen et al. (2002) extend this property to multi-dimensional option problems. The main contribution of these latter extensions is to provide economically satisfactory solutions to such option pricing and hedging problems. Yet, according to Amin (1991), the method proposed by He (1990) are not very useful from a computational perspective. Chen and Yang (1999) constructed a universal trinomial lattice for a large class of diffusion processes. With respect to this branch of research, our contribution is focused on increasing the computational efficiency of the binomial lattice method in a multi-dimensional setting.

In a multi-dimensional setting, Boyle (1988) and Kamrad and Ritchken (1991) provide extensions of their trinomial lattice approach for problems with several underlying assets. Boyle et al. (1989) (BEG, from now on) also provide a straightforward extension of the CRR approach to several underlying assets. However, this approach inherits some unpleasant features of CRR like the possibility of negative probabilities and slow convergence. Ekvall (1996), extending the scheme by Rendleman and Bartter (1979) to a multi-dimensional setting, also proposes a modification (called NEK) of the BEG lattice model to improve convergence and overcome the flaw of possibly negative probabilities.

We herein present a binomial lattice approach for valuing contingent claims dependent on multi-dimensional correlated geometric Brownian processes. The approach relies on two simple ideas: a log-transformation of the value dynamics, as in Trigeorgis (1991); and a change of basis of the asset span to numerically approximate an uncorrelated dynamic for asset
values. This latter idea is taken from what is normally done with Monte Carlo simulation to generate correlated Normal variates. Moreover, we use an additional transformation to eliminate the drift. This further simplifies the numerical scheme.

We provide a wide set of numerical tests proving that the proposed approach is a computational improvement over existing lattice approaches (BEG, NEK and KR). We compare our model to the other lattice approaches without resorting to additional optimizations of the scheme. It would be easy, in order to further improve efficiency to incorporate some of the optimizations proposed in the literature. For instance, if a closed form valuation formula is available, the continuation value at the step just before maturity can be replaced by exact solution, thus increasing accuracy as in the BBS method by Broadie and Detemple (1996). Moreover, Richardson extrapolation or the pruning technique suggested by Baule and Wilkens (2004) can be added also to our scheme.

The proposed method is consistent (i.e., the means and the variance–covariance matrix of the approximating stochastic process are the same as the means and the variance–covariance matrix of the diffusion process for any time step), convergent (i.e., the approximating errors are not amplified), and efficient (i.e., the computational cost for accuracy of a given approximation is lower than in other methods for multi–dimensional options). Interestingly, the relative accuracy and efficiency benefits seem to be greater the larger is the problem dimensionality.

The paper is organized as follows. Section 1 describes the improved binomial lattice approach for approximating multi–dimensional geometric Brownian processes. Section 2 discusses the implementation aspects of our model. In Section 3 we illustrate the efficiency of the proposed approach using several applications to different option pricing problems with up to five stochastic assets. Conclusions are offered in Section 4.

1 The multi–dimensional lattice model

The basic idea is to approximate a multi–dimensional geometric Brownian motion with a binomial lattice after two transformations aiming at preserving the pleasant convergence properties of the one–dimensional additive model. As usual, the first log–transformation permits to approximate an arithmetic Brownian motion. Since the asset returns can be correlated, by changing the coordinate system of the asset span, we can then evaluate an option written on multiple assets by approximating a vector of uncorrelated diffusion processes.

Consider $N$ correlated stochastic non–redundant assets\textsuperscript{3} whose value dy-

\textsuperscript{3}We can always assume that for no pair of assets $i$ and $j$ we have a correlation coefficient $\rho_{ij}$ such that $|\rho_{ij}| = 1$ because, in case this happens, we drop the redundant asset.
namics, denoted $X^\top = (X_1, \ldots, X_N)$, where the symbol $\top$ denotes matrix transposition, follow $N$-dimensional geometric Brownian motions, under the equivalent martingale measure (EMM):\(^4\)

$$
\frac{dX_n}{X_n} = \alpha_n dt + \sigma_n dZ_n \\
X_n(0) = x_n \quad n = 1, 2, \ldots, N
$$

(1)

where $\alpha_n$ is the risk–adjusted drift of the $n$-th asset value,\(^5\) and $dZ_n$ are the increments of correlated Gauss–Wiener processes, such that $E[dZ_i dZ_j] = \rho_{ij} dt$, $i \neq j$, where $\rho_{ij}$ denotes the instantaneous correlation parameter between asset $i$ and asset $j$. We assume that $X$ has a time–independent co-variance matrix. Consider a derivative security with maturity $T$ and value $F$ whose payoff depends on the above underlying asset values. Our goal is to compute $F$. Because in general an analytic solution to this multi-dimensional problem does not exist, one must resort to a numerical solution to approximate $F$. If a lattice method is employed, the solution is found by approximating the continuous–time dynamics in (1) with a discrete–time process convergent in distribution to the continuous–time process by increasing the number of time steps, $M$. Convergence in distribution is a sufficient condition for convergence of the approximated option value to the true option value as long as the payoff is a bounded function of the underlying asset prices. If the payoff is an unbounded function, then, convergence to the actual price is ensured by continuity of the payoff and by uniform integrability of the sequence (as a function of the number of time steps, $M$) of option values.\(^6\)

We use a log–transformed binomial lattice. Taking the logarithm of the asset values,\(^7\) $Y_n = \log X_n / x_n$, the dynamics of $Y^\top = (Y_1, \ldots, Y_N)$ is (by Itô’s Lemma)

$$
dY = adt + \sigma dZ
$$

(2)

\(^4\)For the sake of brevity, we just assume that the conditions ensuring the existence of the EMM hold. For a reference, see for instance Constantinides (1978), Harrison and Kreps (1979), Cox et al. (1985).

\(^5\)If the risk premium, $RP$, is known, $\alpha = g – RP$, where $g$ is the actual growth rate. If the asset is a traded security (or commodity) with a proportional dividend (or, convenience) yield, $\delta$, and $r$ is the risk–free interest rate, $\alpha = r – \delta$. If the asset earns a below–equilibrium rate of return, $\alpha = r – \delta$, where $\delta$ is the rate of return shortfall (see McDonald and Siegel (1984)).

\(^6\)Details are, for instance, in (Duffie, 2001, pp297). Note that in our case, the proof that the sequence (with respect to $M$) of payoffs of the derivative security is uniformly integrable is the same as in the CRR, because we only add a linear transformation to the standard argument.

\(^7\)The idea of approximating the log of the asset value, instead of the asset value itself when it follows a GBM was first introduced in Cox et al. (1979) and then extended to a more general setting by Nelson and Ramaswamy (1990). Note that the discretized scheme we propose is different from the CRR, even if we use the same log–transformation, because ours is designed to be consistent with the continuous process even when $\Delta t$ is finite.
where \( a^\top = (a_1, \ldots, a_N) \), with \( a_n = \alpha_n - \sigma_n^2/2 \), \( dZ^\top = (dZ_1, \ldots, dZ_N) \),

\[
\Sigma = \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1N} \\
\rho_{12} & 1 & \cdots & \rho_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1N} & \rho_{2N} & \cdots & 1
\end{pmatrix}
\quad \text{and} \quad
\sigma = \begin{pmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_N
\end{pmatrix},
\]

where we applied the usual rules: \( dt dZ_n = 0, \ (dt)^2 = 0, \ dZ_i dZ_j = \rho_{ij} dt \).

Next, we transform the basis of the asset span to approximate uncorrelated return dynamics, denoted \( y \). If we change the basis of the market space, we also have to change the payoff function accordingly. Denoting by \( \Pi(Y) \) the original payoff of the option, and by \( W \) the matrix representing the change of basis, the expression of the adjusted option payoff with respect to the new basis is \( \tilde{\Pi}(y) = \Pi(Wy) \). The dynamics of the returns \( y \) can then be approximated by a suitable multi-dimensional binomial lattice. This lattice approach proves to be more efficient than other lattice methods for valuing multi-dimensional options.

The economic rationale for the change of basis is the following. We want to price an option with payoff \( \Pi(Y) \), where \( Y \) are the returns of \( N \) assets traded in the market, in a risk-neutral setting. If the financial markets are complete\(^8\), we can generate \( N \) portfolios with the original assets: we denote \( w_n^\top = (w_{n1}, \ldots, w_{nN}) \) the \( n \)-th portfolio, \( n = 1, \ldots, N \), where \( w_{ij} \) is the amount in the \( j \)-th asset in portfolio \( i \). These portfolios can be thought of as new synthetic assets spanning the (same) market space. Any contingent claim which is spanned by the original assets is spanned also by these synthetic assets. The \( N \) portfolios thus generated are selected so as to have uncorrelated returns. The payoff to be priced, which depends on the returns of these synthetic assets, is denoted \( \tilde{\Pi}(y) \). Because the risk structure of the market is unchanged,\(^9\) we can price the option using risk-neutral valuation with respect to the original EMM by a simple change of basis.

To find the suited change of basis, consider the return dynamics in Equation (2). The covariance matrix of \( dY \) is \( dY dY^\top = \sigma dZ dZ^\top \sigma^\top = \Omega dt \). By definition, \( \Omega \) is a symmetric positive definite matrix. Hence, it can be factorized using an \( N \times N \) matrix \( W \) such that \( WW^\top = I_N \), with \( I_N \) being the \( N \)-dimensional identity matrix, so that \( W^\top \Omega W = \Lambda \), where \( \Lambda \) is the diagonal \( N \)-dimensional matrix \( (\lambda_n) \) with \( \lambda_n > 0 \). We denote by \( y = W^\top Y \) the returns of the synthetic portfolios obtained by linear combinations of the original assets spanning the financial markets. The diffusion process of \( y \) is \( dy = Adt + BdZ \), where \( A = W^\top a \) and \( B = W^\top \sigma \). The covariance

---

\(^8\)The valuation approach remains valid in the more general case of valuing a contingent claim which is redundant with respect to the asset span, according to the EMM.

\(^9\)The market spanned by the synthetic assets is the same as the one where the asset span is generated by the primitive securities. The only difference is how returns are represented.
matrix of $dy$ is $dydy^\top = \Delta dt$, i.e., the components of $y^\top = (y_1, \ldots, y_N)$ are uncorrelated: $dy_idy_j = 0$ whenever $i \neq j$ and $(dy_n)^2 = \lambda_n dt$.

Let $\Pi(X(t)) = \Pi(X_1(t), \ldots, X_N(t))$ be the payoff of the option. According to the change of variable $Y_n = \log X_n/x_n$, the payoff becomes

$$\Pi(x_1e^{Y_1(t)}, \ldots, x_Ne^{Y_N(t)}).$$

We can make the option dependent on $y = W^\top Y$ by changing the payoff function as follows:

$$\tilde{\Pi}(y(t)) = \Pi \left( x_1e^{(Wy(t))_1}, \ldots, x_Ne^{(Wy(t))_N} \right)$$

where $(Wy(t))_n$ is the $n$-th component of $Y(t) = Wy(t)$.

The risk–neutral expected value of the option payoff $\tilde{\Pi}$, denoted $\tilde{F}$, is equal to the risk–neutral expected value of $\Pi$ at the option maturity, $T$.

Hence,

$$\tilde{F}(y(t)) = e^{-r(T-t)}E_y \left[ \tilde{\Pi}(y(T)) \right] = e^{-r(T-t)}E_Y [\Pi(Y(T))] = F(Y(t)), \quad (4)$$

where $E_y[\cdot]$ denotes the risk–neutral expectation with respect to $\nu_y$, the EMM of the process $\{y\}$, and $E_Y[\cdot]$ is the expectation w.r.t. $\nu_Y$, the EMM of the process $\{Y\}$. \(^{10}\) The above is also true for American–type options (see the numerical results in Section 3). \(^{11}\) The intuition of Equation (4) is the following: because the covariance matrix $\Omega$ is time–independent, the measure $\nu_Y$ is invariant under a change of basis. Hence, we can evaluate $\tilde{F}(y)$ by approximating $\nu_y$ with a discrete (binomial) distribution. To do this, we follow the standard procedure used for binomial lattices. Given the option maturity $T$, the time interval $[0, T]$ is divided into $M$ subintervals of increments $\Delta t = T/M$. $M$ is the refinement parameter of the method.

At dates $\{0, \Delta t, 2\Delta t, \ldots, T\}$, the discrete–time approximation of the continuous–time process $\{y\}$ is $\{\tilde{y}^\top\} = \{({\tilde{y}}_1, \ldots, \tilde{y}_N)\}$. For a given $M$, we denote by $\{\tilde{y}^M\}$ the discrete–time approximating process with $M$ time steps. One needs to specify the parameters of the approximating process $\{\tilde{y}^M\}$ so that, as we refine our approximation, $\{\tilde{y}^M\}$ converges in distribution to $\{y\}$.

In what follows, we specify the parameters of the approximating process for a given $M$ (so that the dependence on $M$ will be omitted to simplify

\(^{10}\)Equation (4) can be derived as follows

$$F(Y) = \int e^{-rT}\Pi(Y_T)\nu_Y(dy_T) = \int e^{-rT}\Pi(Wy_T)\nu_Y(Wdy_T)$$

$$= \int e^{-rT}\tilde{\Pi}(y_T)\nu_y(dy_T) = \tilde{F}(y)$$

where $\nu_y(dy) = \nu_Y(Wdy)$ and integration is over the support of $Y(t)$ and $y_T$, respectively.

\(^{11}\)The argument is similar to the one used in Amin and Khanna (1994) for the BEG algorithm.
notation). First, we illustrate the resulting improved binomial scheme for the two-dimensional case. With \( N = 2 \) and \( \rho = \rho_{12} \), we obtain \( \Lambda = (\lambda_n) \), a two-dimensional diagonal matrix, where

\[
\lambda_1 = \frac{1}{2} \left( \frac{\sigma_1^2 + \sigma_2^2 - \sqrt{\sigma_1^4 - 2(1 - 2\rho^2)\sigma_1^2\sigma_2^2 + \sigma_2^4}}{\sigma_1^2} \right),
\]

\[
\lambda_2 = \frac{1}{2} \left( \frac{\sigma_1^2 + \sigma_2^2 + \sqrt{\sigma_1^4 - 2(1 - 2\rho^2)\sigma_1^2\sigma_2^2 + \sigma_2^4}}{\sigma_1^2} \right),
\]

and

\[
W = \left( \begin{array}{c} \frac{\lambda_1 - \sigma_2}{\sigma_1^2} / (\rho c_1) \\ \frac{\lambda_2 - \sigma_1}{\sigma_2^2} / (\rho c_2) \end{array} \right) / 1/c_1
\]

where

\[
c_n = \sqrt{1 + \frac{(\lambda_n - \sigma_n^2)^2}{\rho^2\sigma_1^2\sigma_2^2}}.
\]

The processes of the returns for the synthetic portfolios are

\[
dy_n = A_n dt + B_{n1} dZ_1 + B_{n2} dZ_2 \quad n = 1, 2
\]

where \( B = (B_{ij}) = W^\top \sigma \) and \( A = W^\top a \). We approximate \( \{y\} \) with a discrete process: given the time interval \([0, T]\) and the number of steps \( M \), the discrete approximating process is \( \hat{y} \) with dynamics

\[
\hat{y}_n(t) = \hat{y}_n(t-1) + \ell_n U_n(t) \quad n = 1, \ldots, N
\]

with \( N = 2 \), \( t = 1, \ldots, M \), where \((U_1(t), U_2(t))\) is a set of bi-variate i.i.d. random variables such that

\[
(U_1, U_2) = \begin{cases} 
(1, 1) & \text{with probability } p(1) \\
(1, -1) & \text{with probability } p(2) \\
(-1, 1) & \text{with probability } p(3) \\
(-1, -1) & \text{with probability } p(4)
\end{cases}
\]

where \( \sum_{s=1}^{S} p(s) = 1 \), and \( S = 2^N \) is the number of states at the end of each time step. Convergence in distribution of \( \{\hat{y}\} \) to \( \{y\} \) is equivalent to convergence of the characteristic function of \( \{\hat{y}\} \) to the characteristic function of \( \{y\} \) (see (Billingsley, 1986, pp. 395–396)). Following (Boyle et al., 1989, pp. 245–246), we will determine the parameters of the discrete-time process that match a MacLaurin expansion of the characteristic function of the continuous-time process. The selected parameters are \( \kappa_n = A_n \Delta t \), \( \ell_n = \sqrt{\lambda_n \Delta t + \kappa_n^2} \), \( L_n = \kappa_n / \ell_n \), for \( n = 1, 2 \), and probabilities

\[
p(s) = \frac{1}{4} \left( 1 + \delta_{12}(s)L_1L_2 + \delta_1(s)L_1 + \delta_2(s)L_2 \right) \quad s = 1, \ldots, S,
\]
with $S = 4$ and where
\[
\delta_n(s) = \begin{cases} 
1 & \text{if asset value } n \text{ jumps up} \\
-1 & \text{if asset value } n \text{ jumps down} 
\end{cases} \tag{6}
\]
for $n = 1, 2$, and $\delta_{ij}(s) = \delta_i(s)\delta_j(s)$, $i, j = 1, 2, i \neq j$. With this choice of parameters, the (MacLaurin expansion of the) characteristic function of the discrete distribution matches the (MacLaurin expansion of the) characteristic function of the continuous distribution for any time step:
\[
\begin{align*}
\ell_n \sum_{s=1}^{S} p(s)\delta_n(s) &= A_n \Delta t & n = 1, \ldots, N \\
\ell_n^2 - (A_n \Delta t)^2 &= \lambda_n \Delta t & n = 1, \ldots, N \\
\ell_i \ell_j \sum_{s=1}^{S} p(s)\delta_1(s)\delta_2(s) &= 0 & i \neq j,
\end{align*} \tag{7}
\]
for $N = 2$. Actually, the above specifications of $\kappa_n$, $\ell_n$, $L_n$ and $p(s)$ are found by solving the system of equations in (7).

The above analysis can be generalized to the $N$–dimensional case right away. Concerning the matrices $\Lambda$ and $W$, instead of having exact expressions as in the two–dimensional case, they will be computed numerically (with fair accuracy). Again, for $M$ large enough, we need to match the MacLaurin expansion of the characteristic function of the $N$-dimensional continuous–time distribution with its discrete–time analogue. This is done by solving a system of equations like the one in (7), but for $N \geq 2$. The resulting discrete–time process is $\hat{y}^T = (\hat{y}_1, \ldots, \hat{y}_N)$ with dynamics defined in (5), where $U = (U_1, \ldots, U_N)$ is a set of $N$-variate i.i.d. binomial random variables, with parameters
\[
\begin{align*}
\kappa_n &= A_n \Delta t, & \ell_n &= \sqrt{\lambda_n \Delta t + \kappa_n^2}, & L_n &= \kappa_n / \ell_n, & \text{for } n = 1, \ldots, N, \\
\text{and probabilities} \\
p(s) &= \frac{1}{S} \left( 1 + \sum_{1 \leq i < j \leq N} \delta_{ij}(s)L_i L_j + \sum_{n=1}^{N} \delta_n(s)L_n \right) & s = 1, \ldots, S \tag{8}
\end{align*}
\]
where $S = 2^N$ and $\delta_n(s)$ and $\delta_{ij}(s)$ are defined in (6). The proof is in Appendix A.

Given the process in (5), (8) and (9), the option can be valued applying the discrete–time version of the Bellman equation (for an American–type option)
\[
\tilde{F}(y(t)) = \max \left\{ \Pi(y(t)), e^{-r \Delta t} E_y \left[ \tilde{F}(y(t+1)) \right] \right\}
\]
for $t$ going from $M - 1$ to 0 in a backward dynamic programming fashion.

Summarizing, the proposed binomial lattice approach for multi-dimensional geometric Brownian motion is defined by the parameter choice in equations (8) and (9). We name this *Generalized Log-Transformed* (GLT) approach, because it generalizes the log-transformed approach proposed by Trigeorgis (1991) to a multi-dimensional setting.

In the two-dimensional case, it can be shown that the probabilities of the improved binomial approximation are unconditionally positive and less than one for any parameter values. To see this, observe that Equation (9) for $N = 2$ can be written also as $p(s) = (1 + \delta_1(s)L_1)(1 + \delta_2(s)L_2)/4$. Because $|L_n| < 1$ for all $n$, from Equation (8), $0 < p(s) < 1$ for all $s = 1, \ldots, 4$. When $N > 2$, the GLT approach can have negative probability for some parameter choices.

To have unconditionally positive probabilities, we introduce a variation of the above approach. Since the drift affect probabilities through $L_n$ defined in equation (8), the idea is to approximate drift-less Brownian motions. As it will be shown by numerical tests, this variation provides additional efficiency to the GLT approach.

In details, given the multi-dimensional dynamics in (2), instead of applying the change of basis to $Y$, we first transform the dynamics using position $\bar{Y}_t = Y - at$, so that the process $\bar{Y}$ is drift-less, and then we apply the change of basis $W$ defined above. Note that the variance-covariance matrix for $\bar{Y}$ is the same as for $Y$. Accordingly the payoff becomes

$$\tilde{\Pi}(y(t)) = \Pi \left( x_1 e^{(at+W\nu(t))_1}, \ldots, x_N e^{(at+W\nu(t))_N} \right).$$

Since after this transformation $A = 0$, then from (8) $\kappa_n = 0 = L_n$ for all $n = 1, \ldots, N$. Hence, the proposed approach simplifies to the following approximating process:

$$\hat{y}_n(t) = \hat{y}_n(t - 1) + \sqrt{\lambda_n \Delta t} U_n(t) \quad n = 1, \ldots, N$$

with constant probabilities $p(s) = 1/2^N$, $s = 1, \ldots, S$. We name this variation *Adjusted GLT* or AGLT.

In Section 3 we will assess the efficiency of the proposed lattice methods.

## 2 Implementation procedure

This section discusses implementation of the numerical procedure to value an option according to the multi-dimensional binomial lattice method described in Section 1.

The implementation procedure is similar to the BEG algorithm (see Boyle et al. (1989)) and other proposed binomial lattice algorithms. First,
for a given option maturity $T$ and number of time steps $M$, a multi-
dimensional binomial tree of the evolution of future asset values is gen-
erated according to the parameters in (8) until the time horizon ($T$) is
reached. Note that the routine for diagonalizing the covariance matrix is
called for only once, and so it does not affect the complexity of the algo-

Next, if the option is European, we need only evaluate the option payoff
on the final nodes of the tree (at maturity $T$), average (using the risk–neutral
probabilities determined from (9)) and discount (at the riskless rate) to ob-
tain the current ($t = 0$) value of the option. If the option is American,
we take into account the early exercise feature in a backward dynamic pro-
gramming fashion by comparing, at each node of the lattice, at the indicated
times, the current payoff from early exercise with the option continuation
value obtained by applying the risk–neutral valuation as above.

Discrete dividend–like payments can be accommodated in a standard
way (e.g. Hull and White (1988), and Trigeorgis (1991)). In our model,
discrete non–proportional dividends paid by the underlying assets at known
dates affect the valuation in a similar way as in the Trigeorgis (1991) and
BEG algorithms. The presence of discrete dividends for the $n$-th asset can
be accounted for by a shift of the nodes along the $n$-th dimension at the
ex–dividend date. The displacement is different for each node because the
asset dynamics is exponential, whereas the dividend is additive. Hence,
the tree may not be recombining at the ex–dividend date. In order to
make the tree recombining, the value of the option at the cum–dividend
nodes (i.e., just before the ex–dividend date) can be found by interpolating
the value obtained at the ex–dividend nodes. Given the dividend vector
$D^T = (D_1, \ldots, D_N)$, paid at some known date $\tau$ (the ex–dividend date),
t $\leq \tau < t + \Delta t$, and the value of the option at $t$ from (4),\footnote{We are phrasing this in the GLT approach. Changes are straightforward in the AGLT case.} $\tilde{F}(t, y(t)) = \tilde{F}(t, x_1 e^{(Wy(t))_1}, \ldots, x_N e^{(Wy(t))_N})$, the value of the option at $\tau$ is

\[ \begin{align*}
\tilde{F}(\tau, x_1 e^{(Wy(\tau))_1}, \ldots, x_N e^{(Wy(\tau))_N}) &= \\
&= \tilde{F}(\tau^-, x_1 e^{(Wy(\tau^-))_1} - D_1, \ldots, x_N e^{(Wy(\tau^-))_N} - D_N) + \sum_{n=1}^{N} D_n
\end{align*} \]

where $\tau^-$ is a time just before the ex–dividend date. The main difference
with respect to the BEG algorithm is that the displacement is given accord-
ing to a different (rotated) coordinate set of axes.

The model can be applied also if there is a trigger event for the con-
tingent claim (like a barrier for barrier options) as long as the trigger can
be written in terms of the state variables. Since the applied transformations are one–to–one, if the trigger is $X^* = (X_1^*, \ldots, X_N^*)$, we simply check it in the transformed space by checking the trigger $y^* = W^T Y^*$, where $Y_n^* = \log X_n^*/x_n$, for $n = 1, \ldots, N$.

As suggested by Boyle et al. (1989), Richardson extrapolation (RE) can be a practical method to obtain accurate approximations of exact values while saving on computing time. In particular, we can use a four–point RE, fitting option values (as a function of number of steps $M$) with a cubic polynomial. RE provides accurate estimates of option values as long as the sequence of points is monotonic.

3 Numerical results

In this section we test the performance of the proposed approaches vis–a–vis other lattice approaches proposed for multi–dimensional option problems.

We provide three sets of numerical applications testing the accuracy, the rate of convergence and the efficiency of the GLT (Generalized Log–Transformed) and AGLT (Adjusted GLT) approaches compared to the BEG by Boyle et al. (1989), NEK by Ekvall (1996) and trinomial KR by Kamrad and Ritchken (1991) schemes in 2–, 3– 4–dimensional settings. Moreover, in a 5–dimensional problem, we compare the accuracy of our approach against to the Least–Squares Monte Carlo simulation approach by Longstaff and Schwartz (2001) and simulated trees Broadie and Glasserman (1997).

To test the accuracy of the proposed approach, in dimension three and four, we compare numerical results with exact (analytic) solutions or with published results, in case an analytic solution is missing. Analytic formulae in a multi–dimensional case are available for European call and put options on the maximum and on the minimum of $N$ correlated assets, as introduced by Stulz (1982) and extended by Johnson (1987).\(^\text{13}\) Numerical solutions for the European option on the (arithmetic) average of three assets have been provided by Boyle et al. (1989). The parameters of the problems are chosen in order to meet the valuations made respectively by Boyle et al. (1989) and Ekvall (1996). They are: initial prices, $x_n = 100$; volatilities, $\sigma_n = 0.2$, for $n = 1, 2, 3$; correlations, $\rho_{ij} = 0.5 i \neq j$, $i, j = 1, 2, 3$; risk–free rate $r = 0.1$; maturity, $T = 1$; exercise price, $K = 100$. Table 1 shows the estimates of option values given in Boyle et al. (1989) (improved by RE with $M = 20, 40, 60, 80$) and the relative absolute errors with respect to the accurate value.

\(^\text{13}\)The analytic formulae are available from the authors on request. In these formulae, an efficient algorithm is needed to numerically evaluate the multi–variate cumulative Normal probability. This is by no means a simple task when $N \geq 3$. We greatly benefited from the excellent work by Alan Genz (see Genz (1992, 2004)) and the numerical routines published in his website.
Note that all the numerical option–price estimates obtained by the various binomial lattice approaches (BEG, NEK and GLT and AGLT) converge as the number of steps grows. In terms of accuracy, the proposed approach is generally more accurate than BEG’s scheme. For the options on the max or min, GLT has the same accuracy for as few as 20 time steps as BEG’s approach for more steps. Generally, GLT and AGLT have the same rate of convergence as Ekvall’s NEK model.

Table 2 shows the same results obtained with a four–point Richardson extrapolation using more steps \((M = 40, 80, 120, 160)\). In this case, the NEK and AGLT approaches provide severely flawed numerical estimates because they do not monotonically converge to the true value. This problematic behavior is not observed in the proposed approaches due to a monotonic convergence.

Table 3 replicates Ekvall (1996, Table 2) with European call and put options on the maximum or minimum of four assets, where analytic solutions are available. The parameters of the problem are: initial value, \(x_n = 10\) for \(n = 1, \ldots, 4\); risk–free rate \(r = 0.1\); maturity, \(T = 1\); exercise price, \(K = 10\). Volatilities can be either \(\sigma_n = 0.2\) for all \(n\) or \(\sigma = 0.1\) for all \(n\). Correlations can be either \(\rho_{ij} = 0.1\) for all \(i, j\); or \(\rho_{ij} = 0.7\) for all \(i, j\); or \(\rho_{ij} = 0.5\) for all \(i, j\); or, alternatively \(\rho_{12} = \rho_{13} = \rho_{14} = 0.7\) and \(\rho_{23} = \rho_{24} = \rho = 0.5\). Again, the GLT and AGLT algorithms are as accurate as the NEK approach.

In dimension five, Broadie and Glasserman (1997) (BG, thereafter) provide estimates of an American option on the maximum of five assets. In the same article (BG, Tables 5 and 6), they provide also confidence bounds for the option price using simulated trees with a small number of allowed early exercise dates (4 dates) and a large number (50) of branches. Although they present accurate estimates with two assets (based on Kamrad and Ritchken (1991) trinomial lattice approach), in the five asset case there is no such benchmark and hence no assessment of the relative error is available. A main limitation in BG’s valuation is that it provides downward biased estimates of the American option, since they compute sub–optimal values with restricted early exercise dates.

To provide an appropriate benchmark for assessing the accuracy of our improved binomial model, in Table 4 panel A we present numerical results for the two–dimensional case given in Broadie and Glasserman (1997, Table 3) based on the Least–Squares Monte Carlo simulation\(^{16}\) approach (LSM) proposed by Longstaff and Schwartz (2001). We compare our results to both

\(^{14}\)The only exception is the case of the put option on the average.

\(^{15}\)In fact, their computed confidence range is actually lower than the true confidence range for an American option. This is apparent even (especially) when a Control Variate technique is used: the confidence bounds shrink around a sub–optimal value for the American option (although optimal for a Bermudan option with four early exercise dates).

\(^{16}\)As observed by Longstaff and Schwartz (2001), an approximation error of the continuation value in the LSM algorithm produces a downward–biased option price estimate.
the results provided by LSM and the confidence bounds given by Broadie and Glasserman to establish the accuracy of our method. Then, in Table 4 panel B, we present the five-dimensional case, where there are no known accurate results, comparing our results to the estimates provided by LSM and NEK. Finally, in Figure 1, we show that the proposed model provides accurate estimates for the American option price on five assets with a fairly small number of steps. The five-asset case parameters are: initial asset values, \( x_n = 100 \); dividend yields, \( \delta_n = 0.1 \); volatilities, \( \sigma_n = 0.2 \), for \( n = 1, \ldots, 5 \); correlations, \( \rho_{ij} = 0.3, i \neq j, i, j = 1, \ldots, 5 \); risk-free rate, \( r = 0.05 \); maturity, \( T = 1 \); exercise price, \( K = 100 \). The option payoff at \( t \) is \( \max\{X_1(t) - K, \ldots, X_5(t) - K, 0\} \).

In Table 4A, with two assets, the lattice algorithms give results within the BG confidence bounds.\(^{17}\) LSM simulation also gives results within these bounds, although downward biased. In this specific case, the option estimates provided by GLT and AGLT are close to the values given by BEG and NEK, and the GLT and AGLT algorithms converge faster. In the second part of this section we will see if this behavior is general. For the five-asset case (Table 4B), the results obtained by the three lattice algorithms are almost the same, though sometimes (e.g., \( S_0 = 130 \)) are outside the BG confidence bounds. Figure 1 presents the convergence patterns for the three binomial lattice methods. The values obtained by these lattice methods are always closer to the high estimator than to the low estimator in BG’s bounds. GLT and AGLT provide option estimates very close to the most accurate values with very few steps and very close to each other.\(^{18}\) As in previous cases, NEK exhibits oscillating patterns. The LSM simulation provides downward–biased estimates within BG’s confidence bounds.

In terms of computational cost, the estimates obtained using the proposed methods require less time than the LSM. Though BG confidence bounds can be obtained relatively fast, BG’s algorithm must be applied with a higher number of exercise dates to obtain comparable results and make efficiency comparisons on equal footing.

To test the speed of convergence, in dimension two, three and four, we investigate the behavior of the absolute error

\[
e_M = |\hat{C}_M - C|
\]

where \( C \) is the true (analytic) option value and \( \hat{C}_M \) is the value obtained with a lattice approach, with respect to the refinement parameter (i.e., the number of time steps) \( M \). At the best of our knowledge, a definition of order of convergence has not been given for multi-dimensional lattices. However,

\(^{17}\)This is not the case if the BG confidence bounds are improved with the control variate technique, as can be seen by comparing our results with Broadie and Glasserman (1997, Table 6).

\(^{18}\)With 10 steps the difference from the value obtained with 27 steps is less that 0.1.
applying the same intuition as in Leisen and Reimer (1996) and Leisen (1998) we can empirically derive the order of convergence of a lattice method by graphical inspection.

In the one–dimensional case, the order of convergence of a lattice method is the maximum value of $\beta > 0$ such that $e_M \leq \gamma / M^\beta$ for all possible choices of parameters and for all $M$, for some positive $\gamma$. Although it is beyond the scope of the present work to extend the definition of order of convergence to the multi–dimensional case, we conducted a thorough empirical investigation using a definition of order of convergence which is suited for numerical testing. Given a sample of parameter choices, $\Theta = \{\theta_i, i = 1, \ldots, H\}$, where $\theta_i$ is a vector of parameters for the option problem, and denoted $e_M(\theta)$ the error when the set of parameters is $\theta$, we define the order of convergence as the maximum value of $\beta > 0$ such that the maximum absolute error in the sample is for every $M$ lower than $\gamma / M^\beta$. Formally,

$$\max\{e_M(\theta) \mid \theta \in \Theta\} \leq \gamma / M^\beta \quad \forall M \in \mathbb{N}.$$ 

The numerical analysis is conducted as follows. We consider European call options on the maximum and on the minimum of two, three and four assets. The striking price is held constant, $K = 100$. A sample $\Theta$ of $H = 1500$ vectors of random parameters has been generated in the following hyper–rectangle: $x_n \in [70, 130]$, $\delta_n \in [0, 0.1]$, $\sigma_n \in [0.1, 0.6]$, $\rho_{jk} \in [-0.95, 0.95]$, $r \in [0, 0.1]$, $T \in [0.1, 5]$. Each parameter is selected independently of the others. In generating the sample, we discard the combination of correlations parameters not yielding a positive definite correlation matrix. Moreover, we discard also the random combinations of parameters such that the analytic price of the option is lower than 0.5. Since computations of these plots are cumbersome, we limit our analysis to a small set of values for $M$, but including both even and odd values for $M$ in order to capture the typical oscillating patterns of the absolute error given by lattice methods. In particular, for the 2–dimensional case, we take $M = 7, 14, \ldots, 140$; for the 3–dimensional case, $M = 7, 14, \ldots, 70$; for the 4–dimensional case $M = 7, 14, \ldots, 35$. In addition to the BEG and NEK lattice approaches, here we benchmark the proposed method also to trinomial lattices. In particular, we consider the KR approach applied to a two–, three–, and four–dimensional valuation problem.

Figure 2 shows the results of the numerical experiments and an empirical estimate of the order of convergence. Overall we can say that all the multi–dimensional lattice methods we analyzed have the same order of convergence, which is about one, $\beta \approx 1$. Hence, from this point of view there seems to be no superiority of the proposed lattice methods over existing binomial and trinomial approaches.

Although they are almost indistinguishable as far as the order of convergence is concerned, the lattice approaches proposed in this work are generally
more efficient than the BEG, KR and NEK ones. To show this, we follow
the method proposed by Broadie and Detemple (1996): for a given sample
of valuation problems related to European call options on the maximum and
minimum of two, three and four assets, we analyze the trade–off between
accuracy, in terms of sample average relative error with respect to analytic
solution, and computational speed, measured in number of options prices
computed per second. The sample is selected as previously: while keeping
the striking price constant, \( K = 100 \), \( H = 1500 \) random vectors of parameters
has been generated in the hyper–rectangle: \( x_n \in [70, 130], \delta_n \in [0, 0.1], \sigma_n \in [0.1, 0.6], n = 1, 2, \rho_{jk} \in [-0.95, 0.95], r \in [0, 0.1], T \in [0.1, 5] \). As be-
fore, non valid parameter choices are discarded. Moreover, to make relative
error meaningful, we drop those vectors of parameters yielding \( C < 0.5 \). The
error measure is, with the same notation as above, the Root Mean Square
Error (RMSE):

\[
RMSE = \sqrt{\frac{1}{H} \sum_{i=1}^{H} \left( \frac{\hat{C}_i - C_i}{C_i} \right)^2}.
\]

As usual, since the same hardware has been used, only relative figures mat-
ter.

Figure 3 shows the trade–off between computational speed and accuracy
for the five lattice approaches in the two–, three– and four–dimensional
case. Since the option on the minimum is worth less than the option on
the maximum, the accuracy (measured by relative error) is generally better
for the latter case. In general, AGLT dominates all the other methods. For
instance, in the option on the maximum case, it can be 5–6 times (and
sometimes, as in the 2–dimensional case for the option on the maximum,
about 10 times) faster for the same accuracy. Moreover, in most cases all
methods based on a rotation of the asset span (NEK, GLT and AGLT)
outperform the other lattice approaches.

4 Conclusions

In this work we proposed a binomial lattice approach, called GLT, for valu-
ing contingent claims dependent on multi-dimensional correlated geometric
Brownian motions, which generalizes the approach proposed by Trigeorgis
(1991). The approach relies on two simple ideas: a log–transformation that
is step by step consistent with the continuous–time diffusions, and a change
of basis of the asset span to approximate uncorrelated geometric Brownian
motions. Moreover, we provided a variation of the method based on an
additional transformation to get rid of the drift in the approximating binom-
ial lattice. This further simplifies the numerical scheme (all probabilities

---

\( ^{19} \) The same optimizations of the code have been implemented, in order to put the five
methods on the same footing.
are equal and positive) and proves to be more efficient than other lattice approaches in a multi-dimensional setting. These approaches proves to be consistent, stable and efficient.

Following the methodologies proposed by Broadie and Detemple (1996) and Leisen and Reimer (1996) and Leisen (1998), we provided a thorough documentation of convergence and efficiency of the proposed methods in a two-, three- and four-dimensional setting relative to that of other popular lattice approaches, like the ones proposed by Boyle et al. (1989), Ekvall (1996) and Kamrad and Ritchken (1991). Because no lattice benchmark is available for options on five assets, we compare our results to Broadie and Glasserman (1997) simulated tree algorithm and to Longstaff and Schwartz (2001) Least-Squares Monte Carlo simulation.

While all the lattice methods we analyzed have the same order of convergence, the method we propose dominates in terms of efficiency. Given that our approach entails very simple and intuitive transformations, we think that our result can be helpful both in the financial derivative industry and for financial economic research.
A Derivation of jump sizes and probabilities for the proposed binomial methods

To derive the jump sizes and probabilities for the GLT we follow a well known argument proposed by (Boyle et al., 1989, pp. 245–246) based on convergence of characteristic functions (c.f.) when $\Delta t \to 0$.

With the notation introduced in Section 1, the c.f. of $\Delta y(t) = y(t) - y(t-1)$, where $y^T = (y_1, \ldots, y_N)$ is distributed as a $N$-dimensional Normal with vector of means $A\Delta t$ and covariance matrix $\Lambda\Delta t$, is (dropping the time parameter for simplicity)

$$
\Phi_{\Delta y}(\theta) = \mathbb{E} \left[ \exp \left\{ i \sum_{n=1}^{N} \theta_n \Delta y_n \right\} \right]
$$

$$
= \mathbb{E} \left[ 1 + i \sum_{n=1}^{N} \theta_n \Delta y_n - \frac{1}{2} \left( \sum_{n=1}^{N} \theta_n^2 \Delta y_n^2 + 2 \sum_{j \neq k} \theta_j \theta_k \Delta y_j \Delta y_k \right) + o(\Delta t) \right]
$$

$$
= 1 + i \sum_{n=1}^{N} \theta_n A \Delta t - \frac{1}{2} \sum_{n=1}^{N} \theta_n^2 \left( \lambda_n \Delta t + (A_n \Delta t)^2 \right) + o(\Delta t)
$$

with $\theta^T = (\theta_1, \ldots, \theta_N)$ and where $i$ is the complex unit. In the derivation above, the first equality follows from a MacLaurin expansion and the second equality from the assumption that $y$ is a vector of uncorrelated variates and $\mathbb{E}[\Delta y_n^2] = \lambda_n \Delta t + (A_n \Delta t)^2$.

The c.f. of $\Delta \hat{y}(t) = \hat{y}(t) - \hat{y}(t-1)$, where $\hat{y}^T = (\hat{y}_1, \ldots, \hat{y}_N)$, is (omitting the time parameter and using $\Delta \hat{y}_n = \ell_n \delta_n$, such that for $\Delta t \to 0$, $\ell_n \sim \sqrt{\Delta t}$)

$$
\Phi_{\Delta \hat{y}}(\theta) = \sum_{s=1}^{S} p(s) \exp \left\{ i \sum_{n=1}^{N} \theta_n \ell_n \delta_n(s) \right\}
$$

$$
= \sum_{s=1}^{S} p(s) \left[ 1 + i \sum_{n=1}^{N} \theta_n \ell_n \delta_n(s) - \frac{1}{2} \left( \sum_{n=1}^{N} \theta_n \ell_n \delta_n(s) \right)^2 + o(\Delta t) \right]
$$

$$
= 1 + i \sum_{n=1}^{N} \theta_n \ell_n \sum_{s=1}^{S} p(s) \delta_n(s) - \frac{1}{2} \sum_{n=1}^{N} \theta_n^2 \ell_n^2 - \sum_{j \neq k} \theta_j \theta_k \ell_j \ell_k \sum_{s=1}^{S} p(s) \delta_j(s) \delta_k(s) + o(\Delta t),
$$

where the first equality follows from a MacLaurin expansion; the second equality follows from condition $\sum_{s=1}^{S} p(s) = 1$ and $\delta_n(s)^2 = 1$.

By comparison of the c.f. of $\Delta y$ and of $\Delta \hat{y}$ we have the system of equations in (7). Solving this system we find the parameters $\ell_n$ and $p(s)$ given in Section 1. The same proof applies for the AGLT approach putting $A_n = 0$ for all $n = 1, \ldots, N$. 

19
References


Table 1: European Call and Put Options on the maximum, minimum, and (arithmetic) average of three asset values (see Boyle et al. (1989, Table 2)).

<table>
<thead>
<tr>
<th>Steps</th>
<th>Call</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Put</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BEG</td>
<td>NEK</td>
<td>GLT</td>
<td>AGLT</td>
<td>BEG</td>
<td>NEK</td>
<td>GLT</td>
<td>AGLT</td>
<td></td>
</tr>
<tr>
<td>MAX</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>22.281</td>
<td>22.684</td>
<td>22.685</td>
<td>22.686</td>
<td>0.919</td>
<td>0.929</td>
<td>0.951</td>
<td>0.930</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>22.479</td>
<td>22.686</td>
<td>22.679</td>
<td>22.682</td>
<td>0.925</td>
<td>0.937</td>
<td>0.941</td>
<td>0.931</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>22.544</td>
<td>22.684</td>
<td>22.677</td>
<td>22.679</td>
<td>0.928</td>
<td>0.938</td>
<td>0.938</td>
<td>0.931</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>22.576</td>
<td>22.678</td>
<td>22.676</td>
<td>22.678</td>
<td>0.929</td>
<td>0.934</td>
<td>0.937</td>
<td>0.932</td>
<td></td>
</tr>
<tr>
<td>RE</td>
<td>22.672</td>
<td>22.628</td>
<td>22.673</td>
<td>22.672</td>
<td>0.933</td>
<td>0.889</td>
<td>0.933</td>
<td>0.933</td>
<td></td>
</tr>
<tr>
<td>rel.err.</td>
<td>0.000%</td>
<td>0.194%</td>
<td>0.001%</td>
<td>0.000%</td>
<td>0.001%</td>
<td>4.716%</td>
<td>0.008%</td>
<td>0.033%</td>
<td></td>
</tr>
<tr>
<td>AVG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>22.672</td>
<td>0.933</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MIN</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>5.226</td>
<td>5.235</td>
<td>5.241</td>
<td>5.257</td>
<td>7.240</td>
<td>7.415</td>
<td>7.403</td>
<td>7.431</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>5.237</td>
<td>5.251</td>
<td>5.244</td>
<td>5.249</td>
<td>7.323</td>
<td>7.421</td>
<td>7.405</td>
<td>7.418</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>5.241</td>
<td>5.256</td>
<td>5.245</td>
<td>5.249</td>
<td>7.350</td>
<td>7.422</td>
<td>7.405</td>
<td>7.415</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>5.243</td>
<td>5.249</td>
<td>5.246</td>
<td>5.249</td>
<td>7.364</td>
<td>7.413</td>
<td>7.406</td>
<td>7.413</td>
<td></td>
</tr>
<tr>
<td>rel.err.</td>
<td>0.000%</td>
<td>1.544%</td>
<td>0.005%</td>
<td>0.090%</td>
<td>0.000%</td>
<td>1.095%</td>
<td>0.001%</td>
<td>0.062%</td>
<td></td>
</tr>
<tr>
<td>AVG</td>
<td></td>
<td>5.249</td>
<td></td>
<td></td>
<td>7.406</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AVG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>12.060</td>
<td>12.093</td>
<td>12.062</td>
<td>12.084</td>
<td>2.566</td>
<td>2.577</td>
<td>2.550</td>
<td>2.568</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>12.072</td>
<td>12.088</td>
<td>12.079</td>
<td>12.104</td>
<td>2.567</td>
<td>2.572</td>
<td>2.564</td>
<td>2.588</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>12.076</td>
<td>12.086</td>
<td>12.083</td>
<td>12.074</td>
<td>2.567</td>
<td>2.570</td>
<td>2.568</td>
<td>2.558</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>12.078</td>
<td>12.086</td>
<td>12.084</td>
<td>12.092</td>
<td>2.567</td>
<td>2.570</td>
<td>2.569</td>
<td>2.576</td>
<td></td>
</tr>
<tr>
<td>rel.err.</td>
<td>0.001%</td>
<td>0.018%</td>
<td>0.019%</td>
<td>2.484%</td>
<td>0.006%</td>
<td>0.083%</td>
<td>0.090%</td>
<td>11.692%</td>
<td></td>
</tr>
<tr>
<td>AVG</td>
<td></td>
<td>12.083</td>
<td></td>
<td></td>
<td>2.567</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Case parameters: $X_n(0) = 100$, $\sigma_n = 0.2$, $n = 1, 2, 3$, $\rho_{ij} = 0.5$, $i \neq j$, $i, j = 1, 2, 3$, $r = 0.1$, $T = 1$ and $K = 100$. BEG is the result from Boyle et al. (1989) algorithm; NEK is the result from the algorithm proposed by Ekvall (1996); GLT is from the generalized log-transformed binomial lattice approach and AGLT is from the adjusted GLT proposed in this work. RE = 4-point Richardson extrapolation with $M = 20, 40, 60, 80$ steps. rel.err. = relative error for RE. AV = value from analytic solution for the case of call and put on the maximum and on the minimum of asset prices; value from 4-point Richardson extrapolation with $M = 40, 80, 120, 160$ (see also Table 2) for the case of call and put on the (arithmetic) average of asset prices.
Table 2: European Call and Put Options on the maximum, minimum, and (arithmetic) average of three asset values with Richardson extrapolation.

<table>
<thead>
<tr>
<th>steps</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BEG</td>
<td>NEK</td>
</tr>
<tr>
<td>MAX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>22.479</td>
<td>22.686</td>
</tr>
<tr>
<td>80</td>
<td>22.576</td>
<td>22.678</td>
</tr>
<tr>
<td>120</td>
<td>22.608</td>
<td>22.676</td>
</tr>
<tr>
<td>160</td>
<td>22.624</td>
<td>22.677</td>
</tr>
<tr>
<td>RE</td>
<td>22.672</td>
<td>22.692</td>
</tr>
<tr>
<td>MIN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>5.237</td>
<td>5.251</td>
</tr>
<tr>
<td>80</td>
<td>5.243</td>
<td>5.249</td>
</tr>
<tr>
<td>120</td>
<td>5.245</td>
<td>5.248</td>
</tr>
<tr>
<td>AVG</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>12.072</td>
<td>12.088</td>
</tr>
<tr>
<td>80</td>
<td>12.078</td>
<td>12.086</td>
</tr>
<tr>
<td>120</td>
<td>12.080</td>
<td>12.085</td>
</tr>
<tr>
<td>160</td>
<td>12.081</td>
<td>12.085</td>
</tr>
</tbody>
</table>

Payoffs and parameters are as in Table 1. Here we use more points ($M = 40, 80, 120, 160$) than we did in Table 1. BEG is the result from Boyle et al. (1989) algorithm; NEK is the result from the algorithm proposed by Ekvall (1996) GLT is the estimate from the generalized log-transformed approach. AGLT is the estimate from the adjusted GLT approach. RE = 4-point Richardson extrapolation for $M = 40, 80, 120, 160$. Because the results obtained by NEK approach are oscillatory, Richardson extrapolation is ineffective in some cases (denoted by the symbol *) and numerical valuations are severely flawed.
Table 3: European Call and Put Options on the maximum or minimum of four asset values.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Steps</th>
<th>Call on MAX</th>
<th>Call on MIN</th>
<th>Put on MIN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>BEG</td>
<td>NEK</td>
<td>GLT</td>
</tr>
<tr>
<td>$\sigma_n = 0.2$</td>
<td>10</td>
<td>2.935</td>
<td>2.998</td>
<td>3.007</td>
</tr>
<tr>
<td>$\rho_{ij} = 0.1$ all $i, j$</td>
<td>20</td>
<td>2.973</td>
<td>3.007</td>
<td>3.009</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.992</td>
<td>3.009</td>
<td>3.009</td>
</tr>
</tbody>
</table>

|            |       |     |     |     |     | 3.010 | 0.165 |     | 1.064 |
| $\sigma_n = 0.2$ | 10    | 2.084* | 2.231 | 2.230 | 2.231 | 0.556* | 0.592 | 0.589 | 0.589 | 0.708* | 0.738 | 0.732 | 0.736 |
| $\rho_{ij} = 0.7$ all $i, j$ | 20    | 2.163* | 2.228 | 2.229 | 2.229 | 0.575* | 0.588 | 0.589 | 0.589 | 0.718* | 0.732 | 0.731 | 0.734 |
|            | 40    | 2.197* | 2.228 | 2.228 | 2.228 | 0.582* | 0.589 | 0.589 | 0.589 | 0.724* | 0.732 | 0.731 | 0.733 |

|            |       |     |     |     |     |     |     | 2.227 | 0.589 | 0.731 |
| $\sigma_n = 0.1$ | 10    | 1.620 | 1.702 | 1.701 | 1.702 | 0.418 | 0.448 | 0.453 | 0.449 | 0.195 | 0.213 | 0.214 | 0.213 |
| $\rho_{ij} = 0.5$ all $i, j$ | 20    | 1.663 | 1.701 | 1.700 | 1.701 | 0.434 | 0.447 | 0.451 | 0.449 | 0.203 | 0.210 | 0.213 | 0.212 |
|            | 40    | 1.682 | 1.700 | 1.700 | 1.701 | 0.442 | 0.447 | 0.450 | 0.449 | 0.207 | 0.210 | 0.212 | 0.212 |

|            |       |     |     |     |     | 1.700 | 0.449 |     | 0.211 |
| $\sigma_n = 0.2$ | 10    | 2.247* | 2.365 | 2.382 | 2.372 | 0.466* | 0.493 | 0.486 | 0.492 | 0.762* | 0.791 | 0.792 | 0.793 |
| $\rho_{12} = \rho_{13} = \rho_{14} = 0.7$ | 20    | 2.316* | 2.372 | 2.379 | 2.374 | 0.477* | 0.490 | 0.486 | 0.490 | 0.776* | 0.791 | 0.792 | 0.792 |
| $\rho_{23} = \rho_{24} = \rho_{34} = 0.5$ | 40    | 2.347* | 2.374 | 2.377 | 2.375 | 0.482* | 0.489 | 0.486 | 0.488 | 0.783* | 0.791 | 0.791 | 0.791 |

|            |       |     |     |     |     |     |     | 2.374 | 0.490 | 0.791 |

Valuation of options on four assets see Ekvall (1996, Table 2). Other parameters are $r = 0.1$, $X_n(0) = 10 = K$, $T = 1$. We analyze four combinations of parameters values, as described in the first column.

GLT = estimate given by the generalized log–transformed approach.

AGLT = estimate from the adjusted GLT approach.

AV = analytical value from Ekvall (1996).

* cases with negative jump probabilities in BEG approach.
Table 4: American Call option on the maximum of five (and two) assets

A. American Call option on maximum of two assets

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>LSM</th>
<th>BEG</th>
<th>NEK</th>
<th>GLT</th>
<th>AGLT</th>
<th>BG Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>0.237</td>
<td>0.245</td>
<td>0.243</td>
<td>0.245</td>
<td>0.244</td>
<td>$[0.234,0.263]$</td>
</tr>
<tr>
<td>80</td>
<td>1.259</td>
<td>1.305</td>
<td>1.302</td>
<td>1.305</td>
<td>1.303</td>
<td>$[1.191,1.281]$</td>
</tr>
<tr>
<td>130</td>
<td>36.346</td>
<td>36.453</td>
<td>36.457</td>
<td>36.457</td>
<td>36.457</td>
<td>$[35.221,36.583]$</td>
</tr>
</tbody>
</table>

B. American Call option on maximum of five assets

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>LSM</th>
<th>BEG</th>
<th>NEK</th>
<th>GLT</th>
<th>AGLT</th>
<th>BG Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>0.541</td>
<td>0.544</td>
<td>0.530</td>
<td>0.568</td>
<td>0.554</td>
<td>$[0.536,0.581]$</td>
</tr>
<tr>
<td>80</td>
<td>2.633</td>
<td>2.761</td>
<td>2.724</td>
<td>2.790</td>
<td>2.771</td>
<td>$[2.578,2.746]$</td>
</tr>
<tr>
<td>90</td>
<td>7.678</td>
<td>8.009</td>
<td>7.972</td>
<td>8.019</td>
<td>8.012</td>
<td>$[7.674,8.069]$</td>
</tr>
<tr>
<td>120</td>
<td>36.519</td>
<td>36.984</td>
<td>36.981</td>
<td>37.026</td>
<td>37.026</td>
<td>$[36.121,37.107]$</td>
</tr>
<tr>
<td>130</td>
<td>47.422</td>
<td>48.000</td>
<td>47.992</td>
<td>48.051</td>
<td>48.047</td>
<td>$[46.785,47.888]$</td>
</tr>
</tbody>
</table>

Case parameters: $r = 0.05$, $T = 1$, $K = 100$, $X_n(0) = S_0$, $\sigma_n = 0.2$, $\delta_n = 0.1$ for all $n$, and $\rho_{ij} = 0.3$ for all $i, j$.

The BG bounds in the last column are the 90% confidence intervals of the distribution of the estimate of the option price provided by Broadie and Glasserman (1997).

BEG, GLT, AGLT for Table 5A: $M = 300$ time steps.

BEG, NEK, GLT, AGLT for Table 5B: average of the value obtained with 25 and 26 steps.

We do this because the numerical results for the BEG, NEK and AGLT algorithms oscillate; the GLT algorithm produces a monotonic path towards the (unknown) asymptotic option value.

LSM: estimate of the option value using the Least–Square Monte Carlo simulation algorithm (see Longstaff and Schwartz (2001)) with $M = 50$ time steps and 100 000 paths (50 000 paths for Table 5B); we approximate the continuation value of the option by regressing data on a 5 degree polynomial including all mixed terms up to second degree.
Convergence paths for an American option on the maximum of five assets, Broadie and Glasserman (1997). The straight lines in each plot represents the upper and lower bounds given by Broadie and Glasserman (1997, Table 5) obtained by a simulated tree.

Case parameters: \( X_n(0) = 90, 100, 110 \) respectively, \( \sigma_n = 0.2, \delta_n = 0.1 \) for all \( n \), and \( \rho_{ij} = 0.3 \) for all \( i, j \); \( r = 0.05, T = 1, K = 100 \).
Speed of convergence and empirical examination of the order of convergence for the GLT and the AGLT, compared to the BEG, NEK and KR lattice approaches (log-log scale).

The error here reported is the maximum absolute pricing error of European call options on the maximum and on the minimum of two, three and four assets, for a sample of $H = 1500$ valid vectors of random parameters independently chosen in the hyper-rectangle: $x_n \in [70, 130]$, $\delta_n \in [0, 0.1]$, $\sigma_n \in [0.1, 0.6]$, $n = 1, 2$, $\rho_{jk} \in [-0.95, 0.95]$, $r \in [0, 0.1]$, $T \in [0.1, 5]$. Only combinations of random correlations $\rho_{jk}$ providing a positive definite correlation matrix and an option price $C \geq 0.5$ are considered valid. The striking price is $K = 100$ for all cases. Marks are set for $M = 7, 14, 21, \ldots, 140$ when there are 2 assets; for $M = 7, 14, 21, \ldots, 70$ when there are 3 assets; for $M = 7, 14, \ldots, 35$ when there are 4 assets.

The test function is $\gamma / M^\beta$ with $\beta = 1$ in all cases and $\gamma$ chosen case by case.
Trade–off between speed and accuracy for the GLT and the AGLT, compared to the BEG, NEK and KR lattice approaches. European call option on the maximum and on the minimum of two, three and four correlated assets (log–log scale).

Speed is measured in number of option prices computed per second. The Root Mean Square Error (RMSE) is on a set of values of European call options on the maximum and on the minimum of two, three and four assets. The relative errors refer to a sample of $H = 1500$ valid vectors of random parameters independently chosen in the hyper–rectangle: $x_n \in [70, 130], \delta_n \in [0, 0.1], \sigma_n \in [0.1, 0.6], n = 1, 2, \rho_{jk} \in [-0.95, 0.95], r \in [0, 0.1], T \in [0.1, 5]$. Only combinations of random correlations $\rho_{jk}$ providing a positive definite correlation matrix and an option price $C \geq 0.5$ are considered valid. The striking price is $K = 100$ for all cases.

Marks are set for $M = 20, 40, 60, 80, 120, 140$ when there are 2 assets; for $M = 10, 20, 30, 40, 50, 60, 70$ when there are 3 assets; for $M = 9, 18, 27, 36$ when there are 4 assets.